VIII. *Memoir on the Theory of the Partitions of Numbers.*—Part II.

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Received November 21,—Read November 24, 1898.

Introduction.

Art. 64. The subject of the partition of numbers, for its proper development, requires treatment in a new and more comprehensive manner. The subject-matter of the theory needs enlargement. This will be found to be a necessary consequence of the new method of regarding a partition that is here brought into prominence.

Let an integer n be broken up into any number of integers

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ \alpha_s\,;$$

if we ascribe the conditions

$$\alpha_1 \geqq \alpha_2 \geqq \alpha_3 \geqq \ldots \geqq \alpha_s,$$

the succession

$$\alpha_1,\ \alpha_2,\ \alpha_3\ \ldots\ \alpha_s$$

is what is known as a partition of n.

There are $s - 1$ conditions

$$\alpha_1 \geqq \alpha_2,\ \ \alpha_2 \geqq \alpha_3,\ \ldots\ \alpha_{s-1} \geqq \alpha_s,$$

to which we may add

$$\alpha_s \geqq 0$$

if the integers be all of them positive (or zero). For the present all the integers are restricted to be positive or zero by hypothesis, so that this last-written condition will not be further attended to.

If, on the other hand, the conditions be

$$\alpha_1 \gtrless \alpha_2 \gtrless \alpha_3 \ldots \gtrless \alpha_s,$$

no order of magnitude is supposed to exist between the successive parts, and we obtain what has been termed a "composition" of the integer n.

Various other systems of partitions into s parts may be brought under view, because between two consecutive parts we may place either of the seven symbols

$$>,\ =,\ <,\ \geqq,\ \leqq,\ \gtrless,\ \gtreqless.$$

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We thus obtain 7^{s-1} different sets of conditions that may be assigned; these are not all essentially different and in many cases they overlap.

Art. 65. For the moment I concentrate attention upon the symbol

$$\geqq,$$

and remark that the $s-1$ conditions, which involve this symbol, set forth above, constitute one set of a large class of sets which involve the symbol. We may have the single condition

$$A_1^{(1)}\alpha_1 + A_2^{(1)}\alpha_2 + A_3^{(1)}\alpha_3 + \ldots + A_s^{(1)}\alpha_s \geqq 0,$$

wherein $A_1, A_2, A_3 \ldots A_3$ are integers $+$, zero or $-$, of which at least one must be positive, or we may have the set of conditions

$$\left.\begin{array}{l} A_1^{(1)}\alpha_1 + A_2^{(1)}\alpha_2 + A_3^{(1)}\alpha_3 + \ldots + A_s^{(1)}\alpha_s \geqq 0 \\ A_1^{(2)}\alpha_1 + A_2^{(2)}\alpha_2 + A_3^{(2)}\alpha_3 + \ldots + A_s^{(2)}\alpha_s \geqq 0 \\ A_1^{(3)}\alpha_1 + A_2^{(3)}\alpha_2 + A_3^{(3)}\alpha_3 + \ldots + A_s^{(3)}\alpha_s \geqq 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_1^{(r)}\alpha_1 + A_2^{(r)}\alpha_2 + A_3^{(r)}\alpha_3 + \ldots + A_s^{(r)}\alpha_s \geqq 0 \end{array}\right\}$$

as the definition of the partitions considered. If the symbol be $=$ instead of $\geqq$ the solution of the equations falls into the province of linear Diophantine analysis. The problem before us may be regarded as being one of linear partition analysis. There is much in common between the two theories; the problems may be treated by somewhat similar methods.

The partition analysis of degree higher than the first, like the Diophantine, is of a more recondite nature, and is left for the present out of consideration.

I treat the partition conditions by the method of generating functions. I seek the summation

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \ldots X_s^{\alpha_s}$$

for every set of values (integers)

$$\alpha_1, \ \alpha_2, \ \alpha_3, \ \ldots \ \alpha_s$$

which satisfy the assigned conditions.

It appears that there are, in every case, a finite number of ground or fundamental solutions of the conditions, viz.:—

$$\begin{array}{l} \alpha_1^{(1)}, \ \alpha_2^{(1)}, \ \alpha_3^{(1)} \ldots \alpha_s^{(1)} \\ \alpha_1^{(2)}, \ \alpha_2^{(2)}, \ \alpha_3^{(2)} \ldots \alpha_s^{(2)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_1^{(m)}, \ \alpha_2^{(m)}, \ \alpha_3^{(m)} \ldots \alpha_s^{(m)} \end{array}$$

such that every solution

$$\alpha_1, \ \alpha_2, \ \alpha_3, \ldots \alpha_s$$

is such that

$$\begin{aligned}
\alpha_1 &= \lambda_1\alpha_1^{(1)} + \lambda_2\alpha_1^{(2)} \ldots + \lambda_m\alpha_1^{(m)} \\
\alpha_2 &= \lambda_1\alpha_2^{(1)} + \lambda_2\alpha_2^{(2)} \ldots + \lambda_m\alpha_2^{(m)} \\
\alpha_3 &= \lambda_1\alpha_3^{(1)} + \lambda_2\alpha_3^{(2)} \ldots + \lambda_m\alpha_3^{(m)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha_s &= \lambda_1\alpha_s^{(1)} + \lambda_2\alpha_s^{(2)} \ldots + \lambda_m\alpha_s^{(m}
\end{aligned}$$

$\lambda_1, \lambda_2, \ldots \lambda_m$ being positive integers.

This arises from the fact that every term

$$X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3} \ldots X_s^{\alpha_s}$$

of the summation is found to be expressible as a product

$$\begin{aligned}
&\{X_1^{\alpha_1{}^{(1)}}X_2^{\alpha_2{}^{(1)}}X_1^{\alpha_3{}^{(1)}} \ldots X_s^{\alpha_s{}^{(1)}}\}^{\lambda_1} \\
\times &\{X_1^{\alpha_1{}^{(2)}}X_2^{\alpha_2{}^{(2)}}X_3^{\alpha_3{}^{(2)}} \ldots X_s^{\alpha_s{}^{(2)}}\}^{\lambda_2} \\
\times &\ldots\ldots\ldots\ldots\ldots \\
\times &\{X_1^{\alpha_1{}^{(m)}}X_2^{\alpha_2{}^{(m)}}X_3^{\alpha_3{}^{(m)}} \ldots X_s^{\alpha_s{}^{(m)}}\}^{\lambda_m}
\end{aligned}$$

Denoting this product by

$$P_1^{\lambda_1}P_2^{\lambda_2} \ldots P_m^{\lambda_m}$$

the generating function assumes the form

$$\frac{1 - (Q_1^{(1)} + Q_1^{(2} + Q_1^{(3)} + \ldots) + (Q_2^{(1)} + Q_2^{(2)} + Q_2^{(3)} + \ldots) - (Q_3^{(1)} + \ldots) + \ldots}{(1 - P_1)(1 - P_2)\ldots(1 - P_m)}$$

wherein the denominator indicates the ground solutions and the numerator the simple and compound syzygies which unite them.

The terms

$Q_1^{(1)}, \ Q_1^{(2)}, \ Q_1^{(3)} \ldots$ denote first syzygies
$Q_2^{(1)}, \ Q_2^{(2)}, \ Q_2^{(3)} \ldots$ „ second „
$Q_3^{(1)}, \ Q_3^{(2)}, \ Q_3^{(3)} \ldots$ „ third „
. . . .

The reader will note the striking analogy with the generating functions of the theory of invariants.

Similar results are obtained as solutions of linear Diophantine equations.

The generating functions under view are *real* in the sense of CAYLEY and SYLVESTER. Enumerating generating functions of various kinds are obtained by assigning equalities between the suffixed capitals

$$X_1, X_2, \ldots X_s.$$

Putting, *e.g.*,

$$X_1 = X_2 = \ldots = X_s = x,$$

we obtain the function which enumerates by the coefficient of x^n, in the ascending expansion, the numbers of solutions for which

$$\alpha_1 + \alpha_2 + \ldots + \alpha_3 = n.$$

It will be gathered that the note of the following investigation is the importation of the idea that the solution of any system of equations of the form

$$A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + \ldots + A_s\alpha_s \geqq 0$$

(all the quantities involved being integers) is a problem of partition analysis, and that the theory proceeds *pari passu* with that of the linear Diophantine analysis.

Section 5.

Art. 66. I propose to lead up to the general theory of partition analysis by considering certain simple particular cases in full detail.

Suppose we have a function

$$F(x, a)$$

which can be expanded in ascending powers of x. Such expansion being either finite or infinite, the coefficients of the various powers of x are functions of a which in general involve both positive and negative powers of a. We may reject all terms containing negative powers of a and subsequently put a equal to unity. We thus arrive at a function of x only, which may be represented after CAYLEY (modified by the association with the symbol $\geqq$) by

$$\underset{\geqq}{\Omega} F(x, a),$$

the symbol $\geqq$ denoting that the terms retained are those in which the power of a is $\geqq 0$.

Similarly we may indicate by the operation

$$\underset{=}{\Omega}$$

that the only terms retained are those in which a occurs to the power zero and the meaning of the operations

$$\underset{>}{\Omega},\quad \underset{<}{\Omega},\quad \underset{\leqq}{\Omega},\quad \underset{\gtrless}{\Omega}$$

will be understood without further explanation. To generalise the notion we may consider

$$\underset{\geqq}{\Omega}\, \mathrm{F}\,(\mathrm{X}_1, \mathrm{X}_2, \ldots \mathrm{X}_s, a_1, a_2, \ldots a_t)$$

to mean that the function is to be expanded in ascending powers of $\mathrm{X}_1, \mathrm{X}_2, \ldots \mathrm{X}_s$, the terms involving any negative powers of $a_1, a_2, \ldots a_t$ are to be rejected, and that subsequently we are to put

$$a_1 = a_2 = \ldots = a_t = 1.$$

In this case the operation Ω has reference to each of the letters $a_1, a_2, \ldots a_t$ and a term involving any negative power of either of these quantities is rejected.

If the quantities $a_1, a_2, \ldots a_t$ be not all subjected to the same operation we may denote the whole operation by

$$\underset{\geqq}{\overset{a_1}{\Omega}}\;\underset{>}{\overset{a_2}{\Omega}}\;\underset{\gtrless}{\overset{a_3}{\Omega}}\ldots\ldots\underset{=}{\overset{a_t}{\Omega}}\;\mathrm{F}\,(\mathrm{X}_1, \mathrm{X}_2, \ldots \mathrm{X}_s, a_1, a_2, a_3 \ldots a_t)$$

wherein $\underset{\sigma_r}{\overset{a_r}{\Omega}}$ operates upon a_r according to the law of the symbol σ_r.

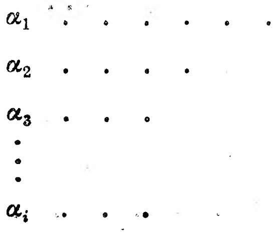
The operation, *quâ* a single quantity and the symbol $\geqq$, have been studied by CAYLEY.* *Quâ* more than one quantity it has presented itself in a memoir on partitions by the present author.†

These Ω functions are of moment in all questions of partition and linear Diophantine analysis.

Art. 67. I will construct Ω functions to serve as generators of well-known solutions and enumerations in the theory of unipartite partition.

Problem I. To determine the number of partitions of w into i or fewer parts.

Graphically considered we have i rows of nodes



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* "On an Algebraical Operation," 'Collected Papers,' vol. 9, p. 537.

† "Memoir on the Theory of the Partitions of Numbers," Part I., 'Phil. Trans.,' A, vol. 187, pp. 619–673, 1896.

$\alpha_1, \alpha_2, \ldots$ denoting the numbers of nodes in the first, second, &c., rows,

$$\begin{gathered}\alpha_1 \geqq \alpha_2\\ \alpha_2 \geqq \alpha_3\\ \vdots\\ \alpha_{i-1} \geqq \alpha_i\end{gathered}$$

To find

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_i^{\alpha_i}$$

for all sets of integers satisfying the conditions take

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1 \,.\, 1 - \frac{a_2}{a_1} X_2 \,.\, 1 - \frac{a_3}{a_2} X_3 \ldots 1 - \frac{a_i}{a_{i-1}} X_i}$$

where observe that the factors $\frac{1}{1 - a_1 X_1}, \frac{1}{1-(a_2/a_1)X_2}, \ldots$ generate the successive rows of nodes and that the method of placing the letters $a_1, a_2, \ldots$ ensures the satisfaction of the first, second, &c., conditions.

Continued application of the simple theorem

$$\underset{\geqq}{\Omega} \frac{1}{1 - ax \,.\, 1 - \frac{1}{a} y} = \frac{1}{1 - x \,.\, 1 - xy},$$

applied in respect of the quantities $a_1, a_2, \ldots$ in succession, reduces the Ω function to the form

$$\frac{1}{1 - X_1 \,.\, 1 - X_1X_2 \,.\, 1 - X_1X_2X_3 \ldots 1 - X_1X_2X_3 \ldots X_i}$$

the *real* generating function.

The ground solutions or fundamental partitions are, as shown by the denominator factors,

$$(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i)$$

$$= \begin{cases} (1, & 0, & 0, & \ldots 0) \\ (1, & 1, & 0, & \ldots 0) \\ (1, & 1, & 1, & \ldots 0) \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \\ (1, & 1, & 1, & \ldots 1) \end{cases}$$

and, as might have been anticipated, the graphical representation is in evidence.

Art. 68. By choosing to sum the expression

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} \dots X_i^{\alpha_i},$$

every solution of the given conditions has been generated. The same result might have been achieved by other summations such as

$$\Sigma X_1^{\lambda_1\alpha_1} X_2^{\mu_2\alpha_2} \dots X_i^{\eta_i\alpha_i},$$

$\lambda_1, \lambda_2, \dots \lambda_i$ being given positive integers, or as

$$\Sigma X_1^{\alpha_1-\alpha_2} X_2^{\alpha_2-\alpha_3} \dots X_{i-1}^{\alpha_{i-1}-\alpha_i} X_i^{\alpha_i}.$$

We, in fact, may take as indices of $X_1, X_2, \dots X_i$ any given linear functions of $\alpha_1, \alpha_2, \dots \alpha_i$, and form the corresponding generating function.

For the two cases specified, the Ω functions are

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1^{\lambda_1} . 1 - \frac{a_2}{a_1} X_2^{\mu_2} \dots 1 - \frac{1}{a_{i-1}} X_i^{\eta_i}},$$

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1 . 1 - \frac{a_2}{a_1}\frac{X_2}{X_1} . 1 - \frac{a_3}{a_2}\frac{X_3}{X_2} \dots 1 - \frac{1}{a_{i-1}}\frac{X_i}{X_{i-1}}},$$

and the reduced functions

$$\frac{1}{1 - X_1^{\lambda_1} . 1 - X_1^{\lambda_1} X_2^{\mu_2} \dots 1 - X_1^{\lambda_1} X_2^{\mu_2} \dots X_i^{\eta_i}},$$

$$\frac{1}{1 - X_1 . 1 - X_2 . 1 - X_3 \dots 1 - X_i}$$

respectively.

Generally for the sum

$$\Sigma X_1^{\lambda_1\alpha_1 + \mu_1\alpha_2 + \dots} X_2^{\lambda_2\alpha_1 + \mu_2\alpha_2 + \dots} \dots X_i^{\lambda_i\alpha_1 + \mu_i\alpha_2 + \dots + \eta_i\alpha_i}$$

the two functions are

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1^{\lambda_1} X_2^{\lambda_2} \dots X_i^{\lambda_i} . 1 - \frac{a_2}{a_1} X_1^{\mu_1} X_2^{\mu_2} \dots X_i^{\mu_i} \dots 1 - \frac{1}{a_{i-1}} X_1^{\eta_1} X_2^{\eta_2} \dots X_i^{\eta_i}}$$

and

$$\frac{1}{1 - X_1^{\lambda_1} X_2^{\lambda_2} \dots X_i^{\lambda_i} . 1 - X_1^{\lambda_1+\mu_1} X_2^{\lambda_2+\mu_2} \dots X_i^{\lambda_i+\mu_i} \dots 1 - X_1^{\lambda_1+\dots+\eta_1} X_2^{\lambda_2+\dots+\eta_2} \dots X_i^{\lambda_i+\dots+\eta_i}}.$$

Art. 69. In any of these instances we have i quantities at disposal, viz.:

$$X_1, X_2, \dots X_i,$$

in order to derive enumerating generating functions corresponding to certain problems. In the last-written general case, the quantities $\lambda, \mu, \ldots \eta$ being given integers, put as a particular case,

$$X_1 = X_2 = \ldots = X_i = x.$$

The reduced function is

$$\frac{1}{1 - x^{\Sigma\lambda} \,.\, 1 - x^{\Sigma\lambda + \Sigma\mu} \,.\, - \ldots 1 - x^{\Sigma\lambda + \Sigma\mu + \ldots + \Sigma\eta}},$$

and herein the coefficients of x^w, in the expansion, give the number of partitions

$$\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_i$$

of all numbers which satisfy the equation

$$\Sigma\lambda \,.\, \alpha_1 + \Sigma\mu \,.\, \alpha_2 + \ldots + \Sigma\eta \,.\, \alpha_i = w,$$

$\alpha_1, \alpha_2, \ldots \alpha_i$ being in descending order.

For the three particular cases considered above this equation takes the forms

$$\alpha_1 + \alpha_2 + \ldots + \alpha_i = w,$$
$$\lambda_1\alpha_1 + \mu_2\alpha_2 + \ldots + \eta_i\alpha_i = w,$$
$$\alpha_1 = w,$$

connected with the reduced generators,

$$\frac{1}{1 - x \,.\, 1 - x^2 \,.\, 1 - x^3 \ldots 1 - x^i},$$

$$\frac{1}{1 - x^{\lambda_1} \,.\, 1 - x^{\lambda_1 + \mu_2} \ldots 1 - x^{\lambda_1 + \mu_2 + \ldots + \eta_i}},$$

$$\frac{1}{(1 - x)^i},$$

respectively.

Further, we may separate $X_1, X_2, \ldots X_i$ in any manner into k sets and put those which are in the first set equal to x_1, those in the second equal to x_2, and so on, and so reach an enumerating function involving k quantities, $x_1, x_2, x_3, \ldots x_k$.

Ex. gr. Put

$$X_1 = X_3 = X_5 = \ldots = x_1,$$
$$X_2 = X_4 = X_6 = \ldots = x_2,$$

and suppose i even. We obtain

$$\frac{1}{1 - x_1 \,.\, 1 - x_1x_2 \,.\, 1 - x_1^2x_2 \,.\, 1 - x_1^2x_2^2 \ldots 1 - x_1^{\frac{1}{2}i}x_2^{\frac{1}{2}i}},$$

to enumerate by the coefficient of $x_1^{w_1}x_2^{w_2}$ those partitions of $w_1 + w_2$ for which

$$\alpha_1 + \alpha_3 + \alpha_5 + \ldots = w_1$$
$$\alpha_2 + \alpha_4 + \alpha_6 + \ldots = w_2.$$

This enumerating function, since it involves x_1 and x_2, is one connected also with the partitions of bipartite numbers. In general when k sets are taken, we have a theorem of k-partite partitions. When $k = i$, we have at once a *real* generating function for unipartites and an enumerating function for i-partites, for, from the latter point of view, the number unity which appears as the coefficient of $X_1^{\alpha_1}X_2^{\alpha_2}\ldots X_i^{\alpha_i}$ shows that the multipartite number

$$\overline{\alpha_1\alpha_2\ldots\alpha_i}$$

can be partitioned in one way only into the parts

$$\overline{1\ 0\ \ldots\ldots\ldots}$$
$$\overline{1\ 1\ 0\ \ldots\ldots}$$
$$\overline{1\ 1\ 1\ \ldots\ldots}$$
$$\ldots\ldots\ldots\ldots$$
$$\overline{1\ 1\ 1\ \ldots\ldots\ 1}$$

there being i figures in each part.

Art. 70. We may now enquire into the partitions of all numbers

$$\alpha_1,\quad \alpha_2,\quad \alpha_3, \ldots \alpha_i,$$

subject to the given conditional relations and also to the linear equations

$$\begin{array}{l}
\lambda_1\alpha_1 + \mu_2\alpha_2 + \ldots + \eta_i\alpha_i = w \\
\lambda'_1\alpha_1 + \mu'_2\alpha_2 + \ldots + \eta'_i\alpha_i = w' \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
\lambda_1^{(s)}\alpha_1 + \mu_2^{(s)}\alpha_2 + \ldots + \eta_i^{(s)}\alpha_i = w^{(s)}.
\end{array}$$

To illustrate the method, it suffices to take $s = 2$, and then we have to perform the summation

$$\Sigma X_1^{\lambda_1\alpha_1}X_2^{\mu_2\alpha_2}\ldots X_i^{\eta_i\alpha_i}Y_1^{\lambda'_1\alpha_1}Y_2^{\mu'_2\alpha_2}\ldots Y_i^{\eta'_i\alpha_i}.$$

The Ω function reduced is

$$\frac{1}{1 - X_1^{\lambda_1}Y_1^{\lambda'_1}\,.\,1 - X_1^{\lambda_1}X_2^{\mu_2}Y_1^{\lambda'_1}Y_2^{\mu'_2}\ldots 1 - X_1^{\lambda_1}X_2^{\mu_2}\ldots X_i^{\eta_i}Y_1^{\lambda'_1}Y_2^{\mu'_2}\ldots Y_i^{\eta'_i}},$$

wherein putting

$$X_1 = X_2 = \ldots = X_i = x,$$
$$Y_1 = Y_2 = \ldots = Y_i = y,$$

we obtain the enumerating function

$$\frac{1}{1 - x^{\lambda_1}y^{\lambda'_1} . 1 - x^{\lambda_1+\mu_2}y^{\lambda'_1+\mu'_2} \ldots 1 - x^{\lambda_1+\mu_2+\ldots+\eta_i}y^{\lambda'_1+\mu'_2+\ldots+\eta'_i}},$$

in which we seek the coefficient of $x^w y^{w'}$.

Art. 71. *Ex. gr.* Consider the particular case

$$\alpha_1 + \alpha_2 + \ldots + \alpha_i = w,$$
$$\alpha_1 + 2\alpha_2 + \ldots + i\alpha_i = w',$$

$\alpha_1, \alpha_2, \ldots \alpha_i$ being, as usual, subject to the conditional relations.

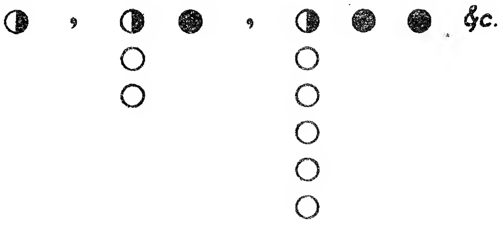
The enumerating function is

$$\frac{1}{1 - xy . 1 - x^2y^3 . 1 - x^3y^6 \ldots 1 - x^iy^{\frac{1}{2}i(i+1)}},$$

and it is obvious also that the partitions of the bipartite $\overline{ww'}$ which satisfy the conditions may be composed by the biparts

$$\overline{11},\ \overline{23},\ \overline{36}, \ldots \overline{i, \tfrac{1}{2}i(i+1)}.$$

The corresponding graphical representation is not by superposition of lines of nodes, but by angles of nodes, of the natures



Art. 72. It is convenient, at this place, to give some elementary theorems concerning the Ω function which will be useful in what follows.

$$\underset{\geq}{\Omega} \frac{1}{1 - ax . 1 - \dfrac{1}{a}y} = \frac{1}{1 - x . 1 - xy},$$

$$\underset{>}{\Omega} \frac{1}{1 - ax . 1 - ay . 1 - \dfrac{1}{a}z} = \frac{1 - xyz}{1 - x . 1 - y . 1 - xz . 1 - yz},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-ax\,.\,1-\frac{1}{a}y\,.\,1-\frac{1}{a}z}=\frac{1}{1-x\,.\,1-xy\,.\,1-xz},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-a^2x\,.\,1-\frac{1}{a}y}=\frac{1+xy}{1-x\,.\,1-xy^2},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-ax\,.\,1-\frac{1}{a^2}y}=\frac{1}{1-x\,.\,1-x^2y},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-a^3x\,.\,1-\frac{1}{a}y}=\frac{1+xy+xy^2}{1-x\,.\,1-xy^3},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-ax\,.\,1-\frac{1}{a^3}y}=\frac{1}{1-x\,.\,1-x^3y},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-a^2x\,.\,1-ay\,.\,1-\frac{1}{a}z}=\frac{1+xz-xyz-xyz^2}{1-x\,.\,1-y\,.\,1-yz\,.\,1-xz^2},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-a^2x\,.\,1-\frac{1}{a}y\,.\,1-\frac{1}{a}z}=\frac{1+xy+xz+xyz}{1-x\,.\,1-xy^2\,.\,1-xz^2},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-ax\,.\,1-ay\,.\,1-az\,.\,1-\frac{1}{a}w}=\frac{1-xyw-xzw-yzw+xyzw+xyzw^2}{1-x\,.\,1-y\,.\,1-z\,.\,1-xw\,.\,1-yw\,.\,1-zw},$$

$$\underset{\geqq}{\Omega}\frac{1}{1-ax\,.\,1-ay\,.\,1-\frac{1}{a}z\,.\,1-\frac{1}{a}}=\frac{1-xyz-xyw-xyzw+xy^2zw+x^2yzw}{1-x\,.\,1-y\,.\,1-xz\,.\,1-xw\,.\,1-yz\,.\,1-yw}.$$

Art. 73. I pass on to consider the partitions of numbers into parts limited not to exceed i in magnitude.

The Ω function is clearly

$$\underset{\geqq}{\Omega}\frac{1-(a_1X_1)^{i+1}}{1-a_1X_1}\,.\,\frac{1-\left(\frac{a_2}{a_1}X_2\right)^{i+1}}{1-\frac{a_2}{a_1}X_2}\,.\,\frac{1-\left(\frac{a_3}{a_2}X_3\right)^{i+1}}{1-\frac{a_3}{a_2}X_3}\;\ldots\;\textit{ad inf.}$$

In this form I have not succeeded in effecting the reduction, but if we put at once

$$X_1=X_2=X_3=\ldots=x,$$

the reduced form is

$$\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3\ldots 1-x^i}$$

If the parts be limited to i in number and to j in magnitude, we find

$$\underset{\geq}{\Omega}\frac{1-(a_1x)^{j+1}}{1-a_1x}\cdot\frac{1-\left(\frac{a_2}{a_1}x\right)^{j+1}}{1-\frac{a_2}{a_1}x}\cdots\frac{1-\left(\frac{1}{a_{i-1}}x\right)^{j+1}}{1-\frac{1}{a_{i-1}}x}=\frac{1-x^{j+1}\,.\,1-x^{j+2}\,.\,1-x^{j+3}\ldots1-x^{j+i}}{1-x\,.\,1-x^2\,.\,1-x^3\ldots1-x^i}$$

the well-known result.

Art. 74. It is to be remarked that the generating function in question may also be written

$$\underset{\geq}{\Omega}\frac{\frac{1}{1-g}}{1-a_1gx\,.\,1-\frac{a_2}{a_1}x\,.\,1-\frac{a_3}{a_2}x^2\ldots1-\frac{1}{a_{i-1}}x},$$

in which we have to seek the coefficient of g^j. This function reduces to

$$\frac{1}{1-g\,.\,1-gx\,.\,1-gx^2\,.\,1-gx^3\ldots1-gx^i}$$

the well-known form.

In general, when a generating function reduces to the product of factors

$$\frac{1}{1-x^s},$$

the part-magnitude being unrestricted, we obtain a product of factors

$$\frac{1}{1-gx^s}$$

for the restricted case, and this is frequently exhibitable, as regards the coefficients of g^j, as a product of factors

$$\frac{1-x^{j+s}}{1-x^s}.$$

The Ω function is not altered by the interchange of the letters i, j.

Art. 75. If the successive parts of the partition are limited in magnitude by

$$j_1,\ j_2,\ \ldots\ j_i,$$

numbers necessarily in descending order, the generating function is

$$\underset{\geq}{\Omega}\frac{1-(a_1x)^{j_1+1}}{1-a_1x}\cdot\frac{1-\left(\frac{a_2}{a_1}x\right)^{j_2+1}}{1-\frac{a_2}{a_1}x}\cdots\frac{1-\left(\frac{1}{a_{i-1}}x\right)^{j_i+1}}{1-\frac{1}{a_{i-1}}x}.$$

For $i = 2$, this may be shown to be equal to

$$\frac{(1 - x^{j_2+1})(1 - x^{j_1+2})}{(1 - x)(1 - x^2)} + x^{j_2+1}\frac{(1 - x^{j_2+1})(1 - x^{j_1-j_2})}{(1 - x)(1 - x^2)},$$

but for $i > 2$, the functions are obtained with increasing labour, and are of increasing complexity.

Many cases present themselves, similar to the one before us, where the Ω function is written down with facility, but no serviceable reduced function appears to exist. On the other hand, we meet with astonishing instances of compact reduced functions which involve valuable theorems.

Art. 76. From the reduced function we can frequently proceed to an $\underset{\geq}{\Omega}$ function, thus inverting the usual process. If, for example, we require an Ω equivalent to

$$\frac{1}{1 - x^{\mathrm{P}_1} \,.\, 1 - x^{\mathrm{P}_2} \,.\, 1 - x^{\mathrm{P}_3} \ldots 1 - x^{\mathrm{P}_i}},$$

a little consideration leads us to

$$\underset{\geq}{\Omega} \frac{1}{1 - a_1 x^{\mathrm{P}_1} \,.\, 1 - \dfrac{a_2}{a_1} x^{\mathrm{P}_2 - \mathrm{P}_1} \,.\, 1 - \dfrac{a_3}{a_2} x^{\mathrm{P}_3 - \mathrm{P}_2} \ldots 1 - \dfrac{1}{a_{i-1}} x^{\mathrm{P}_i - \mathrm{P}_{i-1}}}.$$

This indicates that a unipartite partition into the parts $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_i$ may be represented by a two-dimensional partition of another kind which involves the parts

$$\mathrm{P}_1,\ \mathrm{P}_2 - \mathrm{P}_1,\ \mathrm{P}_3 - \mathrm{P}_2, \ldots \mathrm{P}_i - \mathrm{P}_{i-1}.$$

Ex. gr., the numbers $\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ being in ascending order, the line partition

$$\mathrm{P}_3\mathrm{P}_3\mathrm{P}_3\mathrm{P}_2\mathrm{P}_2\mathrm{P}_1$$

can be thrown into the plane partition

$$\begin{array}{cccccc} \mathrm{P}_1 & \mathrm{P}_1 & \mathrm{P}_1 & \mathrm{P}_1 & \mathrm{P}_1 & \mathrm{P}_1 \\ \mathrm{P}_2 - \mathrm{P}_1 & \mathrm{P}_2 - \mathrm{P}_1 & \mathrm{P}_2 - \mathrm{P}_1 & \mathrm{P}_2 - \mathrm{P}_1 & \mathrm{P}_2 - \mathrm{P}_1 & \\ \mathrm{P}_3 - \mathrm{P}_2 & \mathrm{P}_3 - \mathrm{P}_2 & \mathrm{P}_3 - \mathrm{P}_2. & & & \end{array}$$

of the nature of a regularised graph in the elements $\mathrm{P}_1, \mathrm{P}_2 - \mathrm{P}_1, \mathrm{P}_3 - \mathrm{P}_2$, though these quantities are not necessarily in any specified order of magnitude. We obtain, in fact, a mixed numerical and graphical representation of a partition of a new kind. If

$$(\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3) = (1, 3, 4),$$

the partition 4 3 3 3 1 has the mixed graph

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & \\ 1 & & & & \end{array}$$

as well as its ordinary unit-graph.

In one case the mixed graph is composed entirely of units, and is, moreover, the graph conjugate to the unit graph.

This happens when

$$(P_1, P_2, P_3, \ldots) = (1, 2, 3, \ldots).$$

Thus, *quâ* these elements,

$$4\ 3\ 3\ 3\ 1$$

has the mixed (here the conjugate) graph

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & \\ 1 & & & & \end{array}$$

Art. 77. Observe that a partition may be such *quâ* the parts which actually appear in it, *or* it may be *quâ,* in addition, certain parts which might appear, but which happen to be absent. A mixed graph corresponds to each such supposition.

Ex. gr. :—

Partition.	*Quâ* elements.	Graph.
4 3	4, 3	3 3 1
4 3	4, 3, 1	1 1 2 2 1
4 3	4, 3, 2	2 2 1 1 1
4 3	4, 3, 2, 1	1 1 1 1 1 1 1

We thus arrive at a generalization of the notion of a conjugate partition, and are convinced that the proper representation of a Ferrers-graph is not by nodes or points, but by units.

When the mixed elements

$$P_1,\ P_2 - P_1,\ P_3 - P_2, \ldots$$

are in descending order of magnitude we have a correspondence between unipartite partitions and multipartite partitions of a certain class.

Art. 78. It is usual to consider the parts of a partition arranged in descending order. The Ω function enables us to assign any desired order of magnitude between the successive parts.

In the case of three parts we have already considered the system

$$\alpha_1 \geqq \alpha_2,\ \alpha_2 \geqq \alpha_3.$$

For the system

$$\alpha_1 \geqq \alpha_2,\ \alpha_3 \geqq \alpha_2,$$

we have the solution

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1 . 1 - \frac{1}{a_1 a_2} X_2 . 1 - a_2 X_3},$$

and thence the *real* reduced generator

$$\frac{1}{1 - X_1 . 1 - X_1 X_2 X_3 . 1 - X_3},$$

and the enumerating function

$$\frac{1 + x}{(1 - x)(1 - x^2)(1 - x^3)}.$$

On the other hand, for the system

$$\alpha_2 \geqq \alpha_1,\ \alpha_2 \geqq \alpha_3,$$

we construct

$$\underset{\geqq}{\Omega} \frac{1}{1 - \frac{X_1}{a_1} . 1 - a_1 a_2 X_2 . 1 - \frac{X_3}{a_2}},$$

leading to the real and enumerating functions

$$\frac{1 - X_1 X_2^2 X_3}{1 - X_2 . 1 - X_1 X_2 . 1 - X_2 X_3 . 1 - X_1 X_2 X_3},$$

$$\frac{1 + x^2}{(1 - x)(1 - x^2)(1 - x^3)};$$

of the former, the denominator shows the ground solutions, *id est*, fundamental partitions,

$$(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0);\ (110);\ (011);\ (111);$$

and the enumerator points to the syzygy

$$X_2 . X_1 X_2 X_3 - X_1 X_2 . X_2 X_3 = 0.$$

Art. 79. If the partition be into i parts, we can assign 2^{i-1} different orders depending upon the symbols $\geqq$, $\leqq$, and these can all be expressed by conditional relations affecting $\alpha_1, \alpha_2, \ldots \alpha_i$, involving the symbol $\geqq$ only. These are not all *essentially* different, as one order does or does not give rise to a different order by inversions of parts. Denoting $\geqq$, $\leqq$ by the letters d, a, we have for $i = 3$ the orders dd, da, ad, aa; the orders dd, aa are not essentially different, because interchange of a and d combined with inversion converts the one into the other; da, ad are essentially different, because this two-fold operation leaves each of these unchanged. Hence there are three orders to be considered, and the results have been obtained above.

For $i = 4$ we have the essentially different orders ddd, dda, dad, add. The first of these has been obtained; the other three are solved by the Ω functions:

$$\underset{\geqq}{\Omega}\frac{1}{1 - a_1X_1 \,.\, 1 - \frac{a_2}{a_1}X_2 \,.\, 1 - \frac{X}{a_2a} \,.\, 1 - a_3X_4};$$

$$\underset{\geqq}{\Omega}\frac{1}{1 - a_1X_1 \,.\, 1 - \frac{X_2}{a_1a_2} \,.\, 1 - a_2a_3X_3 \,.\, 1 - \frac{X_4}{a_3}};$$

$$\underset{\geqq}{\Omega}\frac{1}{1 - \frac{X_1}{a_1} \,.\, 1 - a_1a_2X_2 \,.\, 1 - \frac{a_3}{a_3}X_3 \,.\, 1 - \frac{X_4}{a_2}};$$

which reduce to the three expressions:

$$\frac{1}{1 - X_1 \,.\, 1 - X_4 \,.\, 1 - X_1X_2 \,.\, 1 - X_1X_2X_3X_4}$$

$$\frac{1 - X_1X_2X_3^2X_4}{1 - X_1 \,.\, 1 - X_3 \,.\, 1 - X_3X_4 \,.\, 1 - X_1X_2X_3 \,.\, 1 - X_1X_2X_3X}$$

$$\frac{1 - X_1X_2^2X_3 - X_1X_2^2X_3X_4 - X_1X_2^2X_3^2X_4 + X_1X_2^3X_3^2X_4 + X_1^2X_2^3X_3^2X}{1 - X_2 \,.\, 1 - X_1X_2 \,.\, 1 - X_2X_3 \,.\, 1 - X_1X_2X_3 \,.\, 1 - X_2X_3X_4 \,.\, 1 - X_1X_2X_2X};$$

and to the three enumerating functions:

$$\frac{1 + x + x^2}{1 - x \,.\, 1 - x^2 \,.\, 1 - x^3 \,.\, 1 - x^4};$$

$$\frac{1 + x + x^2 + x^3 + x^4}{1 - x \,.\, 1 - x^2 \,.\, 1 - x^3 \,.\, 1 - x^4};$$

$$\frac{1 + x^2 + x^2}{1 - x \,.\, 1 - x^2 \,.\, 1 - x^3 \,.\, 1 - x^4}.$$

The last real generating function that has been written down gives the solution of the system of conditions

$$\alpha_2 \geqq \alpha_1, \quad \alpha_2 \geqq \alpha_3, \quad \alpha_3 \geqq \alpha_4;$$

the ground solutions are

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1, 0, 0), (1, 1, 0, 0), (0, 1, 1, 0), (1, 1, 1, 0), (0, 1, 1, 1), (1, 1, 1, 1);$$

the three simple syzygies are given by

$$\begin{aligned} X_2.X_1X_2X_3 - X_1X_2.X_2X_3 &= S_1 = 0, \\ X_2.X_1X_2X_3X_4 - X_1X_2.X_2X_3X_4 &= S_2 = 0, \\ X_2X_3.X_1X_2X_3X_4 - X_1X_2X_3.X_2X_3X_4 &= S_3 = 0, \end{aligned}$$

and the two compound syzygies by

$$\begin{aligned} X_2X_3X_4.S_1 - X_2X_3.S_2 &= 0, \\ X_1X_2X_3.S_2 - X_1X_2.S_3 &= 0. \end{aligned}$$

Art. 80. In general, when the number of parts is i, we have k_i orders which are altered by interchange of d and a, combined with inversion, and l_i which are unaltered where

$$2k_i + l_i = 2^{i-1}.$$

Hence the number of essentially different orders is

$$k_i + l_i = 2^{i-2} + \tfrac{1}{2}l_i.$$

To determine l_i observe that an order

$$d^{\lambda_1}a^{\mu_1}\, d^{\lambda_2}a^{\mu_2} \ldots d^{\lambda_{s-1}}a^{\mu_{s-1}}\, d^{\lambda_s}a^{\mu_s}$$

will be unaltered by the operations spoken of when

$$\lambda_1 - \mu_s = \mu_1 - \lambda_s = \lambda_2 - \mu_{s-1} = \mu_2 - \lambda_{s-1} = \ldots = 0;$$

so that $i - 1$ must be even and there will be two such unaltered orders for each partition of $i - 1$ into even parts.

Hence the generating function for $k_i + l_i$ is

$$\frac{x^2}{1-2x} + \frac{x}{(1-x^2)(1-x^4)(1-x^6)\ldots ad.\ inf.},$$

giving for

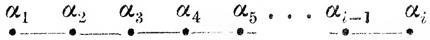
$$i = 2, 3, 4, 5, 6, 7, \ldots$$

the numbers

$$1, 3, 4, 10, 16, 35, \ldots$$

Section 6.

Art. 81. The theory, so far, has been concerned with partitions upon a line. The parts were supposed

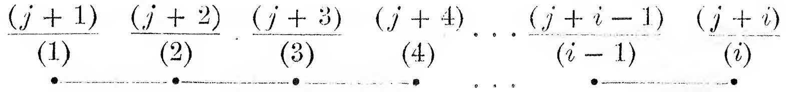


to be placed at the points upon a line with one of the symbols $\geqq$, $\leqq$ placed between every pair of consecutive points.

When the symbol was invariably $\geqq$ the enumerating function found was

$$\frac{(j+1)}{(1)} \cdot \frac{(j+2)}{(2)} \cdot \frac{(j+3)}{(3)} \cdots \frac{(j+i)}{(i)}$$

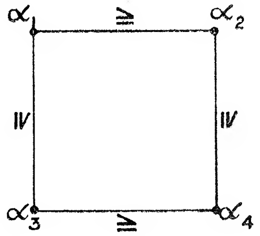
wherein (s) denotes $1 - x^s$. If we place these factors at the successive points of the line we obtain a diagrammatic exhibition of the generating function, viz. :—



a simple fact that the following investigation shows to be fundamental in idea.

Art. 82. I pass on to consider partitions into parts placed at the points of a two-dimensional lattice.

For clearness take the elementary case of four parts placed at the points of a square.



with symbols $\geqq$ placed as shown. We have to solve the conditional relations

$$\alpha_1 \geqq \alpha_2, \qquad \alpha_2 \geqq \alpha_4$$
$$\alpha_1 \geqq \alpha_3, \qquad \alpha_3 \geqq \alpha_4.$$

The four parts are subject to two descending orders. For the sum

$$\Sigma\, X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} X_4^{\alpha_4}$$

we have the Ω function

$$\underset{\geqq}{\Omega}\ \frac{1}{1-abX_1\,.\,1-\frac{d}{a}X_2\ \ 1-\frac{c}{b}X_3\,.\,1-\frac{1}{cd}X_4}$$

which reduces to

$$\frac{1-X_1^2X_2X_3}{1-X_1\,.\,1-X_1X_2\,.\,1-X_1X_3\,.\,1-X_1X_2X_3\,.\,1-X_1X_2X_3X_4},$$

establishing the ground solutions

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 0)\,;\quad (1, 1, 0, 0);\quad (1, 0, 1, 0)\,;\quad (1, 1, 1, 0)\,;\quad (1, 1, 1, 1).$$

connected by the syzygy indicated by

$$X_1\,.\,X_1X_2X_3 - X_1X_2\,.\,X_1X_3 = 0,$$

and leading to the enumerating function

$$\frac{1}{(1-x)\,(1-x^2)^2\,(1-x^3)}.$$

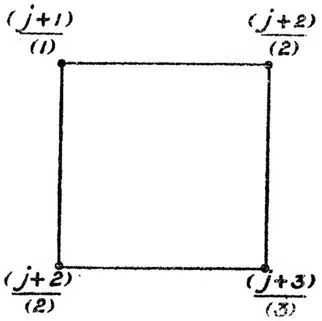
Art. 83. If the parts be restricted not to exceed j in magnitude, we may take as Ω function

$$\underset{\geqq}{\Omega}\ \frac{1-(abX_1)^{j+1}}{1-abX_1\ .\ 1-\frac{d}{a}X_2\ \ 1-\frac{c}{b}X_3\ .\ 1-\frac{1}{cd}X_4},$$

and herein putting $X_1 = X_2 = X_3 = X_4 = x$, and reducing, we get

$$\frac{1-x^{j+1}}{1-x}\ .\ \left(\frac{1-x^{j+2}}{1-x^2}\right)^2\ .\ \frac{1-x^{j+3}}{1-x^3},$$

and we notice that we may represent this diagrammatically on the points of the original lattice, viz. :—



Art. 84. We next have to observe the identity

$$\underset{\geqq}{\Omega}\frac{1}{1-abX_1\,.\,1-\dfrac{d}{a}X_2\,.\,1-\dfrac{c}{b}X_3\,.\,1-\dfrac{1}{cd}X_4}=\underset{\geqq}{\Omega}\frac{1}{1-aX_1\,.\,1-abX_1X_2\,.\,1-\dfrac{1}{a}X_3\,.\,1-\dfrac{1}{ab}X_3X_4},$$

and to note that the dexter leads to the enumerating function

$$\underset{\geqq}{\Omega}\frac{1}{1-ax\,.\,1-abx^2\,.\,1-\dfrac{1}{a}x\,.\,1-\dfrac{1}{ab}x^2},$$

corresponding to the problem of two superposable layers of units, each of two rows;

$$\begin{array}{lll} 1\,1\,1\,1\,1\,1 & \qquad & 1\,1\,1\,1 \\ 1\,1\,1\,1 & & 1\,1 \end{array},$$

in the case indicated superposition yields

$$\begin{array}{l} 2\,2\,2\,2\,1\,1 \\ 2\,2\,1\,1 \end{array};$$

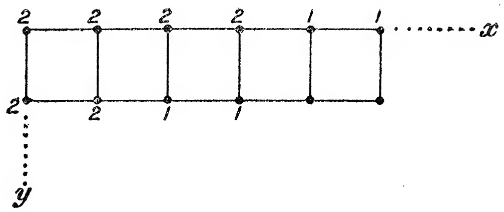
the first row contains a combined number of two's and units $\geqq$ the combined numbers in the second row, and further, the number of two's in first row, $\geqq$ the number of two's in second row. In the Ω function these conditions are secured by the auxiliaries a, b, respectively, and it is established that the problem of partition at the points of the elementary (*i.e.*, simple square) lattice is identical with that of two superposable unit-graphs, each of at most two rows.

In fact, the graph

$$\begin{array}{|l} 2\ \ 2\ \ 2\ \ 2\ \ 1\ \ 1\ \ldots\ldots\ldots\ x \\ 2\ \ 2\ \ 1\ \ 1 \\ \\ \\ \\ \end{array},$$
$$y$$

the axis of z being perpendicular to the plane of the paper, is immediately convertible to the lattice form by projection, with summation of units, upon the plane $y\,z$. The numbers at the points of the square lattice would be 6, 4, 4, 2 respectively.

Art. 85. Observe too that the partition is also one upon another kind of lattice in which the part-magnitude is limited not to exceed 2.



Here, starting from the origin, we may proceed to the opposite point of the lattice along any line of route which proceeds in the positive direction along either axis, and the condition is that along each line of route (here there are six) the numbers must be in descending order and limited in magnitude to 2.

Art. 86. We have, therefore, solved the system of conditions :

$$\begin{array}{ccccccccc} \alpha_1 & \geq & \alpha_2 & \geq & \alpha_3 & \geq & \ldots\ldots & \geq & \alpha_j \\ \mathrm{IV} & & \mathrm{IV} & & \mathrm{IV} & & & & \mathrm{IV} \\ \beta_1 & \geq & \beta_2 & \geq & \beta_3 & \geq & \ldots\ldots & \geq & \beta_j \end{array}$$

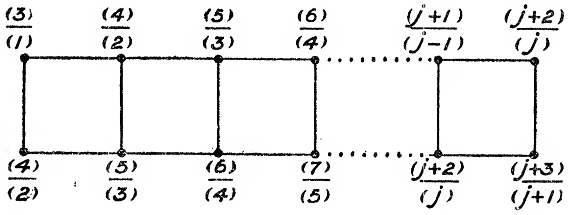
$$2 \geq \alpha_1 \geq 0,$$

which is seen to possess the same solution as the system

$$\begin{array}{ccc} \alpha_1 & \geq & \alpha_2 \\ \mathrm{IV} & & \mathrm{IV} \\ \alpha_3 & \geq & \alpha_4 \end{array}$$

$$j \geq \alpha_1 \geq 0;$$

and we remark the diagrammatic representation



the product of all the factors being

$$\frac{(j+1)\,(j+2)^2\,(j+3)}{(1)\quad(2)^2\quad(3)}.$$

Art. 87. I return to the enumerating function

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)},$$

to note that it may be exhibited as

$$\underset{\geq}{\Omega}\ \frac{1}{1-ax\,.\,1-\frac{b}{a}x\,.\,1-\frac{c}{b}\,.\,1-\frac{1}{c}x};$$

the interpretation of which is that the coefficient of x^w in the development gives the number of instances in which

$$\alpha_1+\alpha_2+\alpha_4=w_1,$$

α_1, α_2, α_3, α_4 being integers satisfying the conditions

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 .$$

We arrive at the form in question if for these conditions we construct

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} X_4^{\alpha_4}$$

and then put $X_1 = X_2 = X_4 = x$.
The graphical representation is of the form

$$\begin{array}{l} 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ldots \\ 1\ 1\ 1\ 1\ 1\ \ldots \\ 0\ 0\ 0\ 0\ \ldots \\ 1\ 1\ \ldots \end{array}$$

the numbers of figures in the rows being in descending order and the third row of figures zeros.

Art. 88. As another instance of the elementary lattice take the system

$$\begin{array}{ll} \alpha_1 \geq \alpha_2, & \alpha_1 \geq \alpha_3 \\ \alpha_4 \geq \alpha_2, & \alpha_4 \geq \alpha_3, \end{array}$$

leading to

$$\underset{\geq}{\Omega}\ \frac{1}{1-abX_1\,.\,1-\frac{1}{ad}X_2 \qquad 1-\frac{1}{bc}X_3\,.\,1-cdX_4},$$

reducing to

$$\frac{1-X_1^2X_2X_3X_4^2}{1-X_1\,.\,1-X_1X_2X_4\,.\,1-X_1X_3X_4\,.\,1-X_1X_2X_3X_4\,.\,1-X_4},$$

establishing the fundamental solutions

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 0)$; $(1, 1, 0, 1)$; $(1, 0, 1, 1)$; $(1, 1, 1, 1,)$; $(0, 0, 0, 1)$;

connected by the syzygy indicated by

$$X_1 \,.\, X_1X_2X_3X_4 \,.\, X_4 - X_1X_2X_4 \,.\, X_1X_3X_4 = 0.$$

Art. 89. A more general generating function connected with the elementary lattice and descending orders is

$$\underset{\geq}{\Omega} \frac{\begin{array}{c} 1 - (abX_1)^{j_1+1} \,.\, 1 - \left(\dfrac{d}{a} X_2\right)^{j_2+1} \\ 1 - \left(\dfrac{c}{b} X_3\right)^{j_3+1} \,.\, 1 - \left(\dfrac{1}{cd} X_4\right)^{j_4+1} \end{array}}{\begin{array}{c} 1 - abX_1 \,.\, 1 - \dfrac{d}{a} X_2 \\ 1 - \dfrac{c}{b} X_3 \,.\, 1 - \dfrac{1}{cd} X_4 \end{array}},$$

where now $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are restricted not to exceed j_1, j_2, j_3, j_4 respectively, and of course

$$\begin{array}{ccc} j_1 & \geq & j_2 \\ \rotatebox[origin=c]{-90}{$\geq$} & & \rotatebox[origin=c]{-90}{$\geq$} \\ j_3 & \geq & j_4 \end{array}$$

are conditions.

It should be remarked that we examine the case of bipartite partitions with regular graphs by putting $X_2 = X_1$, $X_4 = X_3$.

Part-magnitude being unlimited, the reduced function is

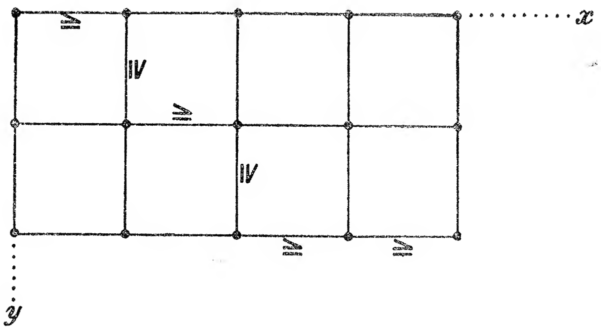
$$\frac{1 - X_1^3X_2}{1 - X_1 \,.\, 1 - X_1^2 \,.\, 1 - X_1X_2 \,.\, 1 - X_1^2X_2 \,.\, 1 - X_1^2X_2^2},$$

and is *real*.

Art. 90. Leaving the particular case, I pass on to consider the general theory of partitions at the points of a lattice in two dimensions. It can be shown immediately that it is coincident with the theory of those partitions of all multipartite numbers which can be represented by regular graphs in three dimensions. For consider the superposition of any number of unit graphs, adding into single numbers the units in the same vertical line. We obtain a scheme of numbers

$$\begin{array}{llllllll} a_{11} & a_{12} & a_{13} & a_{14} & . & . & . & x \\ a_{21} & a_{22} & a_{23} & . & . & . & & \\ a_{31} & a_{32} & . & . & . & & & \\ a_{41} & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ y. & & & & & & & \end{array}$$

in which all the rows and all the columns taken in the positive directions along the axes of x and y are in descending order. We may consider these numbers to be placed at the points of a lattice of which the sides involve m and l points along the sides parallel to the axes of x and y respectively; m will then be a limit to the number of units in any row of a unit graph, and l will be the limit to the number of rows.



There is a descending order along each line of route from the origin to the opposite corner of the lattice, and there are altogether

$$\binom{l+m-2}{l-1} \text{ such lines of route.}$$

Art. 91. The theory of the regular partitions of multipartite numbers is thus reduced to a lattice partition into $l\,m$ parts *in plano*. The conditional relations may be written

$$\begin{array}{ccccccccc}
\alpha_{11} & \geq & \alpha_{12} & \geq & \alpha_{13} & \ldots & \alpha_{1.m-1} & \geq & \alpha_{1m} \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} \\
\alpha_{21} & \geq & \alpha_{22} & \geq & \alpha_{23} & \ldots & \alpha_{2.m-1} & \geq & \alpha_{2.m} \\
 & & & & & & & & \\
\cdot & & \cdot & & \cdot & \ldots & \cdot & & \cdot \\
 & & & & & & & & \\
\alpha_{l-1.1} & \geq & \alpha_{l-1.2} & \geq & \alpha_{l-1.3} & \ldots & \alpha_{l-1.m-1} & \geq & \alpha_{l-1.m} \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} \\
\alpha_{l.1} & \geq & \alpha_{l.2} & \geq & \alpha_{l.3} & \ldots & \alpha_{l.m-1} & \geq & \alpha_{l.m}
\end{array}$$

and for the sum

$$\Sigma \prod_{s=1}^{s=l} \prod_{t=1}^{t=m} X_{st}^{\alpha_{st}}$$

we at once write down the Ω generating function, viz. :—

$$\underset{\geqq}{\Omega}\ \frac{1}{\begin{array}{llll} 1 - a_1\alpha_1 X_{11} & .\ 1 - \frac{a_2}{a_1}\beta_1 X_{12} & .\ 1 - \frac{a_3}{a_2}\gamma_1 X_{13} & \ldots \text{ to } m \text{ factors} \\ 1 - b_1 \frac{\alpha_2}{\alpha_1} X_{21} & .\ 1 - \frac{b_2}{b_1}\frac{\beta_2}{\beta_1} X_{22} & .\ 1 - \frac{b_3}{b_2}\frac{\gamma_2}{\gamma_1} X_{23} & \ldots \text{ to } m \text{ factors} \\ 1 - c_1 \frac{\alpha_3}{\alpha_2} X_{31} & .\ 1 - \frac{c_2}{c_1}\frac{\beta_3}{\beta_2} X_{32} & .\ 1 - \frac{c_3}{c_2}\frac{\gamma_3}{\gamma_2} X_{33} & \ldots \text{ to } m \text{ factors} \\ \vdots & \vdots & \vdots & \&\text{c.} \\ \text{to } l \text{ factors} & \text{to } l \text{ factors} & \text{to } l \text{ factors} & \end{array}}$$

If the part-magnitude be limited to n, we must place as numerator in the function

$$\begin{array}{lll} 1 - \left(a_1\alpha_1 X_{11}\right)^{n+1} & .\ 1 - \left(\frac{a_2}{a_1}\beta_1 X_{12}\right)^{n+1} & \ldots \text{ to } m \text{ factors} \\ 1 - \left(b_1 \frac{\alpha_2}{\alpha_1} X_{21}\right)^{n+1} & .\ 1 - \left(\frac{b_2}{b_1}\frac{\beta_2}{\beta_1} X_{22}\right)^{n+1} & \ldots \text{ to } m \text{ factors} \\ \vdots & \vdots & \&\text{c.} \\ \text{to } l \text{ factors} & \text{to } l \text{ factors} & \end{array}$$

and if we please we may reject all the numerator factors except

$$1 - (a_1\alpha_1 X_{11})^{n+1}.$$

Art. 92. The existence of the three-dimensional graph shows that this function remains unaltered, when X_{st} is put equal to x, for every substitution impressed upon the numbers

$$l,\ m,\ n,$$

but there is a still more refined theorem of reciprocity connected with a more general generating function.

Suppose that the number of layers which involve 1, 2, 3, &c. rows be restricted to

$$l_1,\ l_2,\ l_3,\ \ldots;$$

that the successive layers are restricted to involve at most

$$m_1,\ m_2,\ m_3,\ \ldots \text{ rows};$$

and that the successive rows of the layers are restricted to contain at most

$$n_1,\ n_2,\ n_3,\ \ldots \text{ units}.$$

We have then the comprehensive Ω function :—

$$\underset{\geqq}{\Omega}\frac{\begin{array}{cccl} 1-(a_1\alpha_1X_{11})^{n_1+1} & .\,1-\left(\frac{a_2}{a_1}\beta_1X_{12}\right)^{n_2+1} & .\,1-\left(\frac{a_3}{a_2}\gamma_1X_{13}\right)^{n_3+1} & \text{to } m_1 \text{ factors} \\ 1-\left(b_1\frac{\alpha_2}{\alpha_1}X_{21}\right)^{n_1+1} & .\,1-\left(\frac{b_2}{b_1}\frac{\beta_2}{\beta_1}X_{22}\right)^{n_2+1} & .\,1-\left(\frac{b_3}{b_2}\frac{\gamma_2}{\gamma_1}X_{23}\right)^{n_3+1} & \text{to } m_2 \text{ factors} \\ 1-\left(c_1\frac{\alpha_3}{\alpha_2}X_{31}\right)^{n_1+1} & .\,1-\left(\frac{c_2}{c_1}\frac{\beta_3}{\beta_2}X_{32}\right)^{n_2+1} & .\,1-\left(\frac{c_3}{c_2}\frac{\gamma_3}{\gamma_2}X_{33}\right)^{n_3+1} & \text{to } m_3 \text{ factors} \\ \vdots & \vdots & \vdots & \&\text{c.} \\ \text{to } l_1 \text{ factors} & \text{to } l_2 \text{ factors} & \text{to } l_3 \text{ factors} & \end{array}}{\begin{array}{cccl} 1-a_1\alpha_1X_{11} & .\,1-\frac{a_2}{a_1}\beta_1X_{12} & .\,1-\frac{a_3}{a_2}\gamma_1X_{13} & \ldots \text{ to } m_1 \text{ factors} \\ 1-b_1\frac{\alpha_2}{\alpha_1}X_{21} & .\,1-\frac{b_2}{b_1}\frac{\beta_2}{\beta_1}X_{22} & .\,1-\frac{b_3}{b_2}\frac{\gamma_2}{\gamma_1}X_{23} & \ldots \text{ to } m_2 \text{ factors} \\ 1-c_1\frac{\alpha_3}{\alpha_2}X_{31} & .\,1-\frac{c_2}{c_1}\frac{\beta_3}{\beta_2}X_{32} & .\,1-\frac{c_3}{c_2}\frac{\gamma_3}{\gamma_2}X_{33} & \ldots \text{ to } m_3 \text{ factors} \\ \vdots & \vdots & \vdots & \&\text{c.} \\ \text{to } l_1 \text{ factors} & \text{to } l_2 \text{ factors} & \text{to } l_3 \text{ factors} & \end{array}},$$

wherein, naturally, each of the series

$$\begin{array}{l} l_1, \ l_2, \ l_3, \ \ldots \\ m_1, \ m_2, \ m_3, \ldots \\ n_1, \ n_2, \ n_3, \ \ldots \end{array}$$

is in descending order, and the theorem of reciprocity involved in the fact of the existence of the graph consists in the circumstance that the function remains unaltered, when X_{st} is put equal to x, for any substitution impressed upon the unsuffixed symbols l, m, n.

In the corresponding lattice the conditions are :—

(i.) The first, second, &c., rows do not contain more than n_1, n_2, &c. numbers respectively ;

(ii.) The first, second, &c., rows do not contain higher numbers than l_1, l_2, &c. . . . ;

(iii.) No number so great as s occurs below row m_s for all values of s; $m_1, m_2, \ldots m_s \ldots$ being of course in descending order of magnitude.

Art. 93. The reduction of this Ω function presents great difficulties, and I propose to restrict consideration to the case

$$\begin{array}{l} l_1 = l_2 = l_3 = \ldots = l \\ m_1 = m_2 = m_3 = \ldots = m \\ n_1 = n_2 = n_3 = \ldots = n. \end{array}$$

To adapt the function to enumerate the partitions into at most m parts of l-partite numbers, such partitions being such as possess regular graphs *in solido*, put

$$\begin{aligned} X_{11} &= X_{12} = X_{13} = \ldots = X_{1m} = x_1 \\ X_{21} &= X_{22} = X_{23} = \ldots = X_{2m} = x_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_{l_1} &= X_{l_2} = X_{l_3} = \ldots = X_{l_m} = x_l, \end{aligned}$$

and the resulting function enumerates by the coefficients of

$$x_1^{p_1} x_2^{p_2} \ldots x_l^{p_l},$$

the number of partitions of the l-partite

$$\overline{(p_1 p_2 \ldots p_l)}$$

into at most m parts.

Art. 94. Further putting

$$x_1 = x_2 = x_3 = \ldots = x_l = x,$$

the coefficients of x^w gives the number of graphs *in solido or* unipartite partitions upon a two-dimensional lattice, limited, as indicated above, by the numbers l, m, n.

This function appears to be reducible to the product of factors shown in the tableau below:—

$$\begin{array}{l} \dfrac{(n+1)}{(1)} \cdot \dfrac{(n+2)}{(2)} \cdot \dfrac{(n+3)}{(3)} \ldots \dfrac{(n+m)}{(m)}; \\[2ex] \dfrac{(n+2)}{(2)} \cdot \dfrac{(n+3)}{(3)} \cdot \dfrac{(n+4)}{(4)} \ldots \dfrac{(n+m+1)}{(m+1)}; \\[2ex] \dfrac{(n+3)}{(3)} \cdot \dfrac{(n+4)}{(4)} \cdot \dfrac{(n+5)}{(5)} \ldots \dfrac{(n+m+2)}{(m+2)}; \\ \vdots \\ \dfrac{(n+l)}{(l)} \cdot \dfrac{(n+l+1)}{(l+1)} \cdot \dfrac{(n+l+2)}{(l+2)} \ldots \dfrac{(n+m+l-1)}{(l+m-1)}. \end{array}$$

This result, verified in a multitude of particular cases, awaits demonstration. For $l = 2$ it has been proved independently by Professor **Forsyth** and by the present author. The diagrammatic exhibition of the result at the points of the lattice is clear, and since the product is an invariant for any substitution impressed upon the letters l, m, n, it appears that such exhibition is six-fold. Taking a lattice whose sides contain l m points respectively, so that l m points in all are involved, we mark a corner point, regarding it as an origin of rectangular axes *one*, and proceed to the opposite corner, along any line of route, such that progression along any branch or section of the lattice is in the positive direction, marking the successive points reached *two, three,* &c.

For every point, marked s, we have a factor,

$$\frac{(n+s)}{(s)},$$

and express the generating function as a product of $l\,m$ such factors. If n be ∞, each factor is of the form

$$\frac{1}{(s)},$$

and if the number s appears σ times on the lattice, we have a factor $(s)^{-\sigma}$, and the complete result may be written

$$\frac{1}{(s_1)^{\sigma_1}(s_2)^{\sigma_2}(s_3)^{\sigma_3}\ldots}.$$

Art. 95. Hence the enumeration is identical with that of the partitions of a unipartite number into an unlimited number of parts of $\sigma_1 + \sigma_2 + \sigma_3 + \ldots$ different kinds, viz. :—

σ_1 of the numerical value s_1 but differently coloured.
σ_2 ,, ,, s_2 ,, ,,
σ_3 ,, ,, s_3 ,, ,,
. . . .

The number of distinct lines of route in a lattice of $l\,m$ points is

$$\binom{l+m-2}{l-1},$$

so that, in general, on the lattice we have partitions of a number into $l\,m$ parts subject to $\binom{l+m-2}{l-1}$ descending orders.

Such a partition is transformable ($l \geqq m$) into one composed of the parts

1	of	1	colour
2	,,	2	,,
$\vdots$		$\vdots$	$\vdots$
m	,,	m	,,
$\vdots$		$\vdots$	
l	,,	m	,,
$l+1$	,,	$m-1$	,,
$\vdots$		$\vdots$	
$l+m-1$	,,	1	,,

a theorem of reciprocity analogous to and including the well-known theorem connected with the partitions of a number on a line. There is also a lattice theory connected with unipartite partitions on a line, for the unit-graph of such a partition is nothing more than a number of units and zeros placed at the points of a two-dimensional lattice, such numbers being subject to the $\binom{l+m-2}{l-1}$ descending orders.

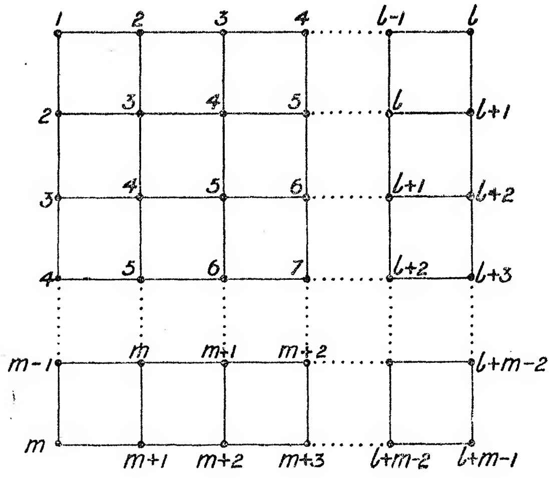
Art. 96. The fact is that the theory of the two-dimensional lattice, the part-magnitude being restricted to unity, is co-extensive with the whole theory of partitions upon a line. Hence for such partitions we may represent the generating function, diagrammatically, in two ways upon a lattice as well as in two ways upon a line.

The two representations upon a line are

$$\frac{(l+1)}{(1)} \;\; \frac{(l+2)}{(2)} \;\; \frac{(l+3)}{(3)} \;\; \frac{(l+4)}{(4)} \;\cdots\; \frac{(l+m-1)}{(m-1)} \;\; \frac{(l+m)}{(m)} .$$

$$\frac{(m+1)}{(1)} \;\; \frac{(m+2)}{(2)} \;\; \frac{(m+3)}{(3)} \;\; \frac{(m+4)}{(4)} \;\cdots\; \frac{(l+m-1)}{(l-1)} \;\; \frac{(l+m)}{(l)} .$$

Upon a lattice we have



and at the point marked s we place the factor

$$\frac{(s+1)}{(s)} .$$

The second lattice is obtained by interchange of l and m.

The product thus obtained is

$$\prod_{s=1}^{s=l+m-1} \left\{ \frac{(s+1)}{(s)} \right\}^{b_s - b_{s-l} - b_{s-m}}$$

b_s denoting the s^{th} figurate number of the second order, and $b_s - b_{s-l} - b_{s-m}$ is easily shown to be equal to the number of points of the lattice marked s. We have to show that this is equal to

$$\prod_{s=1}^{s=m} \frac{(l+s)}{(s)}.$$

Taking $l \geq m$, observe that $(l+s)$ occurs in the former to the power

$$\begin{aligned} &b_{l+s-1} - b_{s-1} - b_{l+s-1-m} \\ - &b_{l+s} + b_s + b_{l+s-m} \end{aligned}$$

which

$$\begin{aligned} &= 1 \text{ if } l + s > m \\ &= 0 \text{ if } l + s \ngtr m; \end{aligned}$$

whilst (s) occurs to the power

$$\begin{aligned} &b_{s-1} - b_{s-1-l} - b_{s-1-m} \\ - &b_s + b_{s-l} + b_{s-m} \end{aligned}$$

which

$$\begin{aligned} &= 1 \quad \text{if} \quad s > m \\ &= 0 \quad \text{if} \quad s > l \text{ and } \ngtr m \\ &= -1 \quad \text{if} \quad s \ngtr l; \end{aligned}$$

the product is, therefore,

$$\frac{\{(l+1)(l+2)\ldots(m)\}\,0\,\{(m+1)(m+2)\ldots(l+m)\}}{\{(1)(2)\ldots(l)\}\,\{(l+1)(l+2)\ldots(m)\}\,0} = \prod_{s=1}^{s=m} \frac{(l+s)}{(s)}.$$

Art. 97. When $l = m = n = \infty$ the generating function is

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^3(1-x^4)^4\ldots},$$

which may be written

$$\underset{\geq}{\Omega} \frac{1}{(1-a_1x)\left(1-\frac{a_2}{a_1}x\,.\,1-\frac{a_3}{a_2}\right)\left(1-\frac{a_4}{a_3}x\,.\,1-\frac{a_5}{a_4}\,.\,1-\frac{a_6}{a_5}\right)(\ldots)\ldots}$$

from which is deduced a graphical representation in two dimensions involving units and zeros.

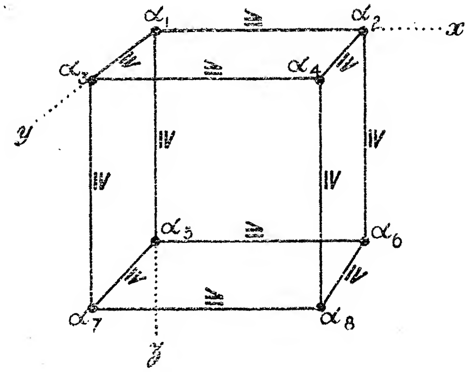
The graph is regular, and the successive rows involve the numbers

1 ; 1, 0 ; 1, 0, 0 ; 1, 0, 0, 0 ; . . .

respectively. In the general case there is a similar representation, proper restrictions being placed upon the numbers of figures in the rows.

Section 7.

Art. 98. It might have been conjectured that the lattice *in solido* would have afforded results of equal interest, but this on investigation does not appear to be the case. The simplest of such lattices is that in which the points are the summits of a cube and the branches the edges of the cube.



$\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\ \alpha_8$ is a partition of a number into eight parts, satisfying the conditional relations indicated by the symbols $\geqq$ as shown. The descending order is in the positive direction parallel to each axis. The Ω function

$$\underset{\geqq}{\Omega} \frac{1}{\begin{array}{l} 1 - a_1a_2a_3 X_1 \,.\, 1 - \dfrac{a_4a_5}{a_1} X_2 \,.\, 1 - \dfrac{a_6a_7}{a_2} X_3 \,.\, 1 - \dfrac{a_8a_9}{a_3} X_4 \\ 1 - \dfrac{a_{10}}{a_4a_6} X_5 \,.\, 1 - \dfrac{a_{11}}{a_7a} X_6 \,.\, 1 - \dfrac{a_{12}}{a_5a_8} X_7 \,.\, 1 - \dfrac{1}{a_{10}a_{11}a_{12}} X_8 \end{array}}$$

is difficult to deal with, and the result which I have obtained too complicated to be worth preserving. I therefore put at once

$$X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = x,$$

and seek the sum $\Sigma x^{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8}$. I divide the calculation into eighteen parts as follows :—

Conditions.	Result.
$\alpha_6 \geqq \alpha_7 \geqq \alpha_4$ $\alpha_5 \geqq \alpha_2, \quad \alpha_5 \geqq \alpha_3$	$\dfrac{1 + x^3 + x^4}{(1)\,(2)\,(3)\,(4)\,(5)\,(6)\,(7)\,(8)}$
$\alpha_6 \geqq \alpha_7 \geqq \alpha_4$ $\alpha_2 \geqq \alpha_3, \quad \alpha_2 > \alpha_5$	$\dfrac{x^2 + x^3 + x^6}{(1)\,(2)\,(3)\,(4)\,(5)\,(6)\,(7)\,(8)}$

Conditions.	Result.
$\alpha_6 \geqq \alpha_7 \geqq \alpha_4$ $\alpha_3 > \alpha_2, \quad \alpha_3 > \alpha_5$	$\dfrac{x^2 + x^5}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_6 \geqq \alpha_4, \quad \alpha_4 > \alpha_7$ $\alpha_5 \geqq \alpha_2, \quad \alpha_5 \geqq \alpha_3$	$\dfrac{x^6 + x^9 + x^{10}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_6 \geqq \alpha_4, \quad \alpha_4 > \alpha_7$ $\alpha_2 \geqq \alpha_3, \quad \alpha_2 > \alpha_5$	$\dfrac{x^8 + x^9 + x^{12}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_6 \geqq \alpha_4, \quad \alpha_4 > \alpha_7$ $\alpha_3 > \alpha_2, \quad \alpha_3 > \alpha_5$	$\dfrac{x^8 + x^{11}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_4 > \alpha_6, \quad \alpha_6 \geqq \alpha_7$ $\alpha_5 \geqq \alpha_2, \quad \alpha_5 \geqq \alpha_3$	$\dfrac{x^5 + x^8}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_4 > \alpha_6, \quad \alpha_6 \geqq \alpha_7$ $\alpha_2 \geqq \alpha_3, \quad \alpha_2 > \alpha_5$	$\dfrac{x^4 + x^7 + x^8}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_4 > \alpha_6, \quad \alpha_6 \geqq \alpha_7$ $\alpha_3 > \alpha_2, \quad \alpha_3 > \alpha_5$	$\dfrac{x^6 + x^7 + x^{10}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_4 > \alpha_7, \quad \alpha_7 > \alpha_6$ $\alpha_5 \geqq \alpha_2, \quad \alpha_5 \geqq \alpha_3$	$\dfrac{x^{11} + x^{14}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_4 > \alpha_7, \quad \alpha_7 > \alpha_6$ $\alpha_2 \geqq \alpha_3, \quad \alpha_2 > \alpha_5$	$\dfrac{x^{10} + x^{13} + x^{14}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_4 > \alpha_7, \quad \alpha_7 > \alpha_6$ $\alpha_3 > \alpha_2, \quad \alpha_3 > \alpha_5$	$\dfrac{x^{12} + x^{13} + x^{16}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_7 \geqq \alpha_4 > \alpha_6$ $\alpha_5 \geqq \alpha_2, \quad \alpha_5 \geqq \alpha_3$	$\dfrac{x^6 + x^9 + x^{10}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_7 \geqq \alpha_4 > \alpha_6$ $\alpha_2 \geqq \alpha_3, \quad \alpha_2 > \alpha_5$	$\dfrac{x^8 + x^9}{(1)(2)(3)(4)(5)(6)(7)(8)}$

Conditions.	Result.
$\alpha_7 \geqq \alpha_4 > \alpha_6$ $\alpha_3 > \alpha_2, \quad \alpha_3 > \alpha_5$	$\dfrac{x^8 + x^{11} + x^{12}}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_7 > \alpha_6 \geqq \alpha_4$ $\alpha_5 \geqq \alpha_2, \quad \alpha_5 \geqq \alpha_3$	$\dfrac{x^4 + x^5 + x^8}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_7 > \alpha_6 \geqq \alpha_4$ $\alpha_2 \geqq \alpha_3, \quad \alpha_2 > \alpha_5$	$\dfrac{x^7 + x^8}{(1)(2)(3)(4)(5)(6)(7)(8)}$
$\alpha_7 > \alpha_6 \geqq \alpha_4$ $\alpha_3 > \alpha_2, \quad \alpha_3 > \alpha_5$	$\dfrac{x^6 + x^7 + x^{10}}{(1)(2)(3)(4)(5)(6)(7)(8)}$

and by addition the resulting generating function* is

$$\frac{1 + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 5x^6 + 4x^7 + 8x^8 + 4x^9 + 5x^{10} + 3x^{11} + 3x^{12} + 2x^{13} + 2x^{14} + x^{16}}{(1)(2)(3)(4)(5)(6)(7)(8)}.$$

Art. 99. By analogy with the lattice *in plano* one might have conjectured that the result would have been

$$\frac{1}{(1)(2)^3(3)^3(4)};$$

but this is not so, although the two functions do coincide as far as the coefficient of x^5 inclusive. In fact, the two expansions yield respectively

$$1 + x + 4x^2 + 7x^3 + 14x^4 + 23x^5 + 41x^6 + 63x^7 + \ldots,$$
$$1 + x + 4x^2 + 7x^3 + 14x^4 + 23x^5 + 42x^6 + 63x^7 + \ldots,$$

the succeeding coefficients becoming widely divergent. This at first seemed surprising, but observe that analogy might also lead us to expect that, if the part-magnitude be limited to i, the result would be

$$\frac{(i+1)(i+2)^3(i+3)^3(i+4)}{(1)(2)^3(3)^3(4)};$$

but this does not happen to be expressible in a finite integral form for all values of i, a fact which necessitates the immediate rejection of the conjecture. The expression in question is only finite and integral when i is of the form $3p$ or $3p + 1$. We have, further, the fact that the expression does give the enumeration when $i = 1$, for then the generating function is easily ascertainable to be

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* Mr. A. B. KEMPE, Treas. R.S., has verified this conclusion by a different and most ingenious method of summation, which also readily yields the result for any desired restriction on the part-magnitude.

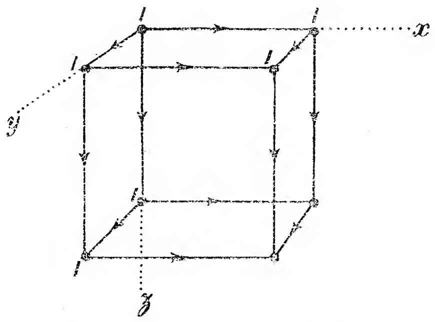
$$1 + x + 3x^2 + 3x^3 + 4x^4 + 3x^5 + 3x^6 + x^7 + x^8,$$

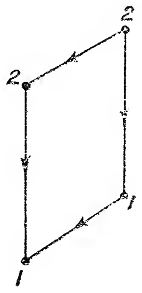
which may be exhibited in the forms

$$\frac{(4)^2\,(5)}{(1)\,(2)^2} \equiv \frac{(3)\,(4)^2\,(5)}{(1)\,(2)^2\,(3)} \equiv \frac{(2)\,(3)^3\,(4)^3\,(5)}{(1)\,(2)^3\,(3)^3\,(4)}.$$

Art. 100. The second of these forms immediately arrests the attention, for, *in plano*, it denotes the number of partitions on a lattice of four points (in fact, a square), the part-magnitude being limited not to exceed 2. The reason of this is as follows :—

Taking the cube with any distribution of units at the summits, we may project the summits upon the plane of $y\,z$, adding up the units on the cube edges at right angles to that plane, and thus obtain a distribution, on the points of the cube face in that plane, of numbers limited in magnitude to 2.





This projection establishes the theorem, which may now be generalized. Conceive a lattice *in solido* having l, m, n points along the axes of x, y, z respectively, and a distribution of units at the points of the lattice which form an unbroken succession along each line of route through the lattice from the origin to the opposite corner, a line of route always proceeding parallel to the axes in a positive sense. Now project and sum units on the plane of $y\,z$.

The result is a partition of the number at the points of a lattice *in plano* whose sides contain m and n points respectively, the part-magnitude being limited not to exceed l. The descending order in this lattice is clearly from the origin to the opposite corner in the plane $y\,z$ along each of its lines of route.

The enumerating generating function is

$$\begin{array}{rccccc}
 & \frac{(l+1)}{(1)} \,. & \frac{(l+2)}{(2)} \,. & \frac{(l+3)}{(3)} & \ldots\ldots & \frac{(l+m)}{(m)} \\
\times & \frac{(l+2)}{(2)} \,. & \frac{(l+3)}{(3)} \,. & \frac{(l+4)}{(4)} & \ldots\ldots & \frac{(l+m+1)}{(m+1)} \\
\times & \frac{(l+3)}{(3)} \,. & \frac{(l+4)}{(4)} \,. & \frac{(l+5)}{(5)} & \ldots\ldots & \frac{(l+m+2)}{(m+2)} \\
 & \vdots & \vdots & \vdots & & \vdots \\
\times & \frac{(l+n)}{(n)} \,. & \frac{(l+n+1)}{(n+1)} \,. & \frac{(l+n+2)}{(n+2)} & \ldots & \frac{(l+m+n)}{(m+n)} \,.
\end{array}$$

Each factor may be supposed at a point of the corresponding lattice ; if any point is the s^{th} along a line of route the factor is

$$\frac{(l+s)}{(s)}\,.$$

The number of points at which we place

$$\frac{(l+s)}{(s)}$$

is equal to the coefficient of x^s in the expansion of

$$x\,(1+x+x^2+\ldots+x^{m-1})\,(1+x+x^2+\ldots+x^{n-1})$$

that is of

$$\frac{x}{(1+x)^2}\,(1-x^m)\,(1-x^n).$$

If m, n be in ascending order and b_s denote the s^{th} figurate number of the second order, this coefficient is

$$b_s - b_{(s-m)} - b_{(s-n)}$$

the term $+\,b_{s-m-n}$ being omitted because s is at most $m+n-1$.

Hence the generating function may be written

$$\prod_{s=1}^{s=m+n-1}\left\{\frac{(l+s)}{(s)}\right\}^{b_s-b_{s-m}-b_{s-n}}.$$

Art. 101. It is now important to show the connexion between this result and the original lattice *in solido*.

I say that this generating function may be exhibited by factors placed at the points of the lattice *in solido*. These factors are of form

$$\frac{(s+1)}{(s)},$$

and such a factor must be placed at every point which is the s^{th} occurring along a line of route in the cubic reticulation.

I take l, m, n in ascending order, and remark that the number of points possessing this property is the coefficient of x^s in the product

$$x\,(1 + x + x^2 + \ldots + x^{l-1})\;(1 + x + x^2 + \ldots + x^{m-1})\;(1 + x + x^2 + \ldots + x^{n-1}),$$

which is

$$\frac{x}{(1-x)^3}\,(1-x^l)\,(1-x^m)\,(1-x^n),$$

and that, if c_s denote the s^{th} of the third order of figurate numbers, this coefficient is

$$c_s - c_{s-l} - c_{s-m} - c_{s-n} + c_{s-l-m} + c_{s-l-n} + c_{s-m-n},$$

the term $-\,c_{s-l-m-n}$ being omitted, because s is at most $l + m + n - 2$.

I propose, therefore, to prove the identity

$$\prod_{s=1}^{s=m+n-1}\left\{\frac{(l+s)}{(s)}\right\}^{b_s - b_{s-m} - b_{s-n}} = \prod_{s=1}^{s=l+m+n-2}\left\{\frac{(s+1)}{(s)}\right\}^{c_s - c_{s-1} - c_{s-m} - c_{s-n} + c_{s-1-m} + c_{s-1-n} + c_{s-m-n}}$$

The factor $(l + s)$ occurs to the power

$$-\,b_{l+s} + b_s - b_{s-m} - b_{s-n} + b_{l+s-m} + b_{l+s-n}$$

on the sinister side, and to the power

$$\begin{aligned}&-(c_{l+s} - c_{l+s-1}) + (c_s - c_{s-1}) - (c_{s-m} - c_{s-m-1})\\&-(c_{s-n} - c_{s-n-1}) + (c_{l+s-m} - c_{l+s-m-1}) + (c_{l+s-n} - c_{l+s-n-1})\end{aligned}$$

on the dexter. But

$$c_k - c_{k-1} = b_k = k.$$

Hence, under all circumstances, the two powers must be equal.

Again the factor (s) occurs to the power,

$$-\,b_s + b_{s-l} + b_{s-m} + b_{s-n} - b_{s-l-m} - b_{s-l-n}$$

on the sinister side, and to the power

$$-(c_s - c_{s-1}) + (c_{s-l} + c_{s-l-1}) + (c_{s-m} - c_{s-m-1})$$
$$+ (c_{s-n} - c_{s-n-1}) - (c_{s-l-m} - c_{s-l-m-1}) - (c_{s-l-n} - c_{s-l-n-1})$$

on the dexter, and again the two powers are equal.

Hence the identity under consideration is established, and this carries with it the proof of the diagrammatic representation of the generating function on the points of the solid reticulation.

Art. 102. I resume the general theory of the partitions on the summits of a cube. When the parts are unrestricted in magnitude the generating function has been found. A process similar to that employed leads to the theorem that when the parts are restricted not to exceed t in magnitude the generating function is the quotient of

$$\begin{aligned} 1 &+ a(2x^2 + 2x^3 + 3x^4 + 2x^5 + 2x^6) \\ &+ a^2(x^5 + 3x^6 + 4x^7 + 8x^8 + 4x^9 + 3x^{10} + x^{11}) \\ &+ a^3(2x^{10} + 2x^{11} + 3x^{12} + 2x^{13} + 2x^{14}) \\ &+ a^4 . x^{16} \end{aligned}$$

by

$$(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)(1-ax^5)(1-ax^6)(1-ax^7)(1-ax^8),$$

the required number being given by the coefficient of a^tx^w. Denoting the numerator by $1 + a\text{P}(x) + a^2\text{Q}(x) + a^3\text{R}(x) + a^4 . x^{16}$, the whole coefficient of a^t is

$$\frac{(9)(10)\ldots(t+8)}{(1)(2)\ldots(t)} + \text{P}(x)\frac{(9)(10)\ldots(t+7)}{(1)(2)\ldots(t-1)} + \text{Q}(x)\frac{(9)(10)\ldots(t+6)}{(1)(2)\ldots(t-2)}$$
$$+ \text{R}(x)\frac{(9)(10)\ldots(t+5)}{(1)(2)\ldots(t-3)} + x^{16} . \frac{(9)(10)\ldots(t+4)}{(1)(2)\ldots(t-4)} .$$

Denoting this generating function by $\text{F}_t(x)$, I find

$$\text{P}(x) = \text{F}_1(x) - \frac{(9)}{(1)},$$

$$\text{Q}(x) = \text{F}_2(x) - \frac{(9)}{(1)}\text{F}_1(x) + x\frac{(8)(9)}{(1)(2)},$$

$$\text{R}(x) = \text{F}_3(x) - \frac{(9)}{(1)}\text{F}_2(x) + x\frac{(8)(9)}{(1)(2)}\text{F}_1(x) - x^3\frac{(7)(8)(9)}{(1)(2)(3)},$$

$$x_{16} = \text{F}_4(x) - \frac{(9)}{(1)}\text{F}_3(x) + x\frac{(8)(9)}{(1)(2)}\text{F}_2(x) - x^3\frac{(7)(8)(9)}{(1)(2)(3)}\text{F}_1(x) + x^6\frac{(6)(7)(8)(9)}{(1)(2)(3)(4)},$$

whence

$$\text{F}_5(x) = \frac{(9)}{(1)}\text{F}_4(x) - x\frac{(8)(9)}{(1)(2)}\text{F}_3(x) + x^3\frac{(7)(8)(9)}{(1)(2)(3)}\text{F}_2(x)$$
$$- x^6\frac{(6)(7)(8)(9)}{(1)(2)(3)(4)}\text{F}_1(x) + x^{10} . \frac{(5)(6)(7)(8)(9)}{(1)(2)(3)(4)(5)},$$

and in general

$$\begin{aligned} F_t(x) = {} & \frac{(9)(10)\ldots(t+4)}{(1)(2)\ldots(t-4)} F_4(x) - x\frac{(8)(9)\ldots(t+4)}{(1)(2)\ldots(t-3)} \cdot \frac{(t-4)}{(1)} F_3(x) \\ & + x^3 \frac{(7)(8)\ldots(t+4)}{(1)(2)\ldots(t-2)} \cdot \frac{(t-4)(t-3)}{(1)(2)} \cdot F_2(x) \\ & - x^6 \frac{(6)(7)\ldots(t+4)}{(1)(2)\ldots(t-1)} \cdot \frac{(t-4)(t-3)(t-2)}{(1)(2)(3)} F_1(x) \\ & + x^{10} \frac{(5)(6)\ldots(t+4)}{(1)(2)\ldots(t)} \cdot \frac{(t-4)(t-3)(t-2)(t-1)}{(1)(2)(3)(4)} . \end{aligned}$$

Art. 103. This appears to be the most symmetrical form in which the generating function can be exhibited, and it may be assumed that the like function for the solid reticulation in general will be of complicated nature. The argument that has been given shows that the theory of the n-dimensional lattice (easily realizable *in plano*), the part-magnitude being limited so as not to exceed unity, is co-extensive with the whole theory of partitions on the lattice of $n - 1$ dimensions.

Section 8.

Art. 104. The enumerating generating functions that are met with at the outset in the theory of the partitions of numbers are such as are formed by factors of the forms

$$\frac{1 - x^{n+s}}{1 - x^s},$$

written for brevity $\frac{(n+s)}{(s)}$. All those which appear in connection with regular graphs in two and three dimensions are so expressible, and the mere fact of such expression proves beyond question that the numerator of the generating function is exactly divisible by the denominator; in other words, it proves that the function can be put into a finite integral form. It is quite natural therefore to seek the general expression of functions of this form, which possesses this property of competency to generate a finite number of terms. Moreover, it is conceivable that such a determination will indicate the paths of future research in these matters: will be in fact a sign-post at the cross-ways. This is the reason why I undertook the investigation; but, as frequently happens in similar cases, the problem proves *à posteriori* to be *per se* of great interest and to involve in itself a notable theorem in partitions.

Art. 105. I consider the function

$$\frac{(n+1)^{\alpha_1}(n+2)^{\alpha_2}(n+3)^{\alpha_3}\ldots(n+s)^{\alpha_s}}{(1)^{\alpha_1}\quad(2)^{\alpha_2}\quad(3)^{\alpha_3}\quad\ldots\quad(s)^{\alpha_s}},$$

which I also write

$$X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\ldots X_s^{\alpha_s},$$

and investigate the sum

$$\Sigma X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \dots X_s^{\alpha_s}$$

for all values of $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_s$, which render the expression under the sign of summation expressible in a finite integral form *for all values of the integer n.*

Art. 106. Let ξ_t be that factor of $1 - x^t$ which, when equated to zero, yields all the primitive roots of the equation

$$1 - x^t = 0.$$

Then $1 - x^t = \xi_1 \xi_{d_1} \xi_{d_2} \dots \xi_t$ where $1, d_1, d_2, \dots t$ are all the divisors of t. We must find the circumstances under which every expression ξ_t will occur at least as often in the numerator as in the denominator. We need not attend to ξ_1, since it occurs with equal frequency in numerator and denominator. In regard to ξ_2, we have equal frequency if $n + 1$ be uneven, but if $n + 1$ be even we must have

$$\alpha_1 + \alpha_3 + \alpha_5 + \dots \geqq \alpha_2 + \alpha_4 + \alpha_6 + \dots$$

For ξ_3 if $n + 1 \equiv 0 \bmod 3$,

$$\alpha_1 + \alpha_4 + \alpha_7 + \dots \geqq \alpha_3 + \alpha_6 + \alpha_9 + \dots,$$

and if $n + 1 \equiv 1 \bmod 3$,

$$\alpha_2 + \alpha_5 + \alpha_8 + \dots \geqq \alpha_3 + \alpha_6 + \alpha_9 + \dots,$$

while the case of $n + 1 \equiv 2 \bmod 3$ need not be attended to.

Proceeding in this manner we find the following conditions :—

$$\begin{array}{l}
\quad \alpha_1 + \alpha_3 + \alpha_5 + \dots \geqq \alpha_2 + \alpha_4 + \alpha_6 + \dots \\
\left\{\begin{array}{l}
\alpha_1 + \alpha_4 + \alpha_7 + \dots \geqq \alpha_3 + \alpha_6 + \alpha_9 + \dots \\
\alpha_2 + \alpha_5 + \alpha_8 + \dots \geqq \alpha_3 + \alpha_6 + \alpha_9 + \dots
\end{array}\right. \\
\left\{\begin{array}{l}
\alpha_1 + \alpha_5 + \alpha_9 + \dots \geqq \alpha_4 + \alpha_8 + \alpha_{12} + \dots \\
\alpha_2 + \alpha_6 + \alpha_{10} + \dots \geqq \alpha_4 + \alpha_8 + \alpha_{12} + \dots \\
\alpha_3 + \alpha_7 + \alpha_{11} + \dots \geqq \alpha_4 + \alpha_8 + \alpha_{12} + \dots
\end{array}\right. \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\left\{\begin{array}{ll}
\alpha_1 + \alpha_s + \dots\dots & \geqq \alpha_{s-1} \\
\alpha_2 & \geqq \alpha_{s-1} \\
\vdots & \vdots \\
\alpha_{s-2} & \geqq \alpha_{s-1}
\end{array}\right. \\
\left\{\begin{array}{ll}
\alpha_1 & \geqq \alpha_s \\
\alpha_2 & \geqq \alpha_s \\
\vdots & \vdots \\
\alpha_{s-1} & \geqq \alpha_s
\end{array}\right.
\end{array}$$

$\frac{1}{2} s (s - 1)$ in number.

The next step is to construct an Ω function which shall express these conditions and lead practically to the desired summation.

Art. 107. First take $s = 2$; there is but one condition

$$\alpha_1 \geqq \alpha_2,$$

and the function is

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1 . 1 - \frac{1}{a_1} X_2} = \frac{1}{1 - X_1 . 1 - X_1 X_2},$$

and every term in the ascending expansion of this function is of the required form, and no other forms exist. The general term being

$$X_1^{\alpha_1 - \alpha_2} (X_1 X_2)^{\alpha_2} \qquad \alpha_1 \geqq \alpha_2,$$

we may call X_1 and $X_1 X_2$ the ground forms from which all other forms are derived.

Art. 108. Next take $s = 3$. The conditions are

$$\left.\begin{array}{l} \alpha_1 + \alpha_3 \geqq \alpha_2 \\ \alpha_1 \qquad \geqq \alpha_3 \\ \alpha_2 \qquad \geqq \alpha_3 \end{array}\right\},$$

leading to the summation formula

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 a_2 X_1 . 1 - \frac{a_3}{a_1} X_2 . 1 - \frac{a_1}{a_2 a_3} X},$$

the auxiliaries a_1, a_2, a_3 determining the first, second and third conditions respectively. The function is equal to

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1 a_2 X_1 . 1 - \frac{1}{a_1} X_2 . 1 - \frac{1}{a_2} X_2 X_3}$$

$$= \underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1 . 1 - \frac{1}{a_1} X_2 . 1 - a_1 X_1 X_2 X_3}$$

$$= \underset{\geqq}{\Omega} \frac{1}{1 - a_1 X_1} \left\{ \frac{1}{1 - a_1 X_1 X_2 X_3 . 1 - X_1 X_2^2 X_3} + \frac{\frac{X_2}{a_1}}{1 - \frac{X_2}{a_1} . 1 - X_1 X_2^2 X_3} \right\}$$

$$= \frac{1}{1 - X_1 . 1 - X_1 X_2 X_3 . 1 - X_1 X_2^2 X_3} + \frac{X_1 X_2}{1 - X_1 . 1 - X_1 X_2 . 1 - X_1 X_2^2 X_3}$$

$$= \frac{1 - X_1^2 X_2^2 X_3}{1 - X_1 . 1 - X_1 X_2 . 1 - X_1 X_2 X_3 . 1 - X_1 X_2^2 X_3},$$

representing the complete solution.

The denominator factors yield the ground forms

$$X_1X_2X_3, \qquad X_1X_2^2X_3$$

in addition to those previously met with, whilst the numerator factor indicates the ground form syzygy

$$X_1 \,.\, X_1X_2^2X_3 - X_1X_2 \,.\, X_1X_2X_3 = 0.$$

Observe that

$$X_1X_2X_3 = \frac{1 - x^{n+1} \,.\, 1 - x^{n+2} \,.\, 1 - x^{n+3}}{1 - x \,.\, 1 - x^2 \,.\, 1 - x^3}$$

$$X_1X_2^2X_3 = \frac{(1 - x^{n+1})(1 - x^{n+2})^2(1 - x^{n+3})}{(1 - x)(1 - x^2)^2(1 - x^3)}$$

are those with which we are familiar in the theories of simple and compound partition respectively.

Art. 109. I pass on to the case $s = 4$; the conditions are

$$\begin{aligned} \alpha_1 + \alpha_3 &\geqq \alpha_2 + \alpha_4 \\ \alpha_1 + \alpha_4 &\geqq \alpha_3 \\ \alpha_2 \quad\; &\geqq \alpha_3 \\ \alpha_1 \quad\; &\geqq \alpha_4 \\ \alpha_2 \quad\; &\geqq \alpha_4 \\ \alpha_3 \quad\; &\geqq \alpha_4 \end{aligned}$$

We neglect the fifth of these as being implied by the remainder and from the function

$$\underset{\geqq}{\Omega} \frac{1}{1 - a_1a_2a_4X_1 \,.\, 1 - \dfrac{a_3}{a_1}X_2 \,.\, 1 - \dfrac{a_1a_5}{a_2a_3}X_3 \,.\, 1 - \dfrac{a_2}{a_1a_4a_5}X_4}$$

which, when reduced, is

$$\frac{1}{1 - X_1 \,.\, 1 - X_1X_2 \,.\, 1 - X_1X_2X_3X_4 \,.\, 1 - X_1X_2^2X_3^2X_4}$$

$$+ \frac{X_1X_2^2X_3}{1 - X_1 \,.\, 1 - X_1X_2 \,.\, 1 - X_1X_2^2X_3 \,.\, 1 - X_1X_2^2X_3^2X_4}$$

$$+ \frac{X_1X_2X_3}{1 - X_1 \,.\, 1 - X_1X_2X_3 \,.\, 1 - X_1X_2^2X_3 \,.\, 1 - X_1X_2^2X_3^2X_4}$$

showing that the new ground forms are $X_1X_2X_3X_4$ and $X_1X_2^2X_3^2X_4$, both of which have presented themselves before.

The result may be written

$$\frac{1 - X_1^2X_2^2X_3 - X_1^2X_2^2X_3^2X_4 - X_1^2X_2^3X_3^2X_4^1 + X_1^3X_2^3X_3^2X_4 + X_1^3X_2^4X_3^3X_4}{1 - X_1 \,.\, 1 - X_1X_2 \,.\, 1 - X_1X_2X_3 \,.\, 1 - X_1X_2^2X_3 \,.\, 1 - X_1X_2X_3X_4 \,.\, 1 - X_1X_2^2X_3^2X_4};$$

and the numerator now indicates the existence of first and second syzygies between the ground forms.

We have the first syzygies

$$\begin{aligned}(\mathrm{A}) &= \mathrm{X}_1\mathrm{X}_2 \,.\, \mathrm{X}_1\mathrm{X}_2\mathrm{X}_3 - \mathrm{X}_1 \,.\, \mathrm{X}_1\mathrm{X}_2^2\mathrm{X}_3 = 0,\\ (\mathrm{B}_1) &= \mathrm{X}_1\mathrm{X}_2\mathrm{X}_3 \,.\, \mathrm{X}_1\mathrm{X}_2\mathrm{X}_3\mathrm{X}_4 - \mathrm{X}_1 \,.\, \mathrm{X}_1\mathrm{X}_2^2\mathrm{X}_3^2\mathrm{X}_4 = 0,\\ (\mathrm{B}_2) &= \mathrm{X}_1\mathrm{X}_2^2\mathrm{X}_3 \,.\, \mathrm{X}_1\mathrm{X}_2\mathrm{X}_3\mathrm{X}_4 - \mathrm{X}_1\mathrm{X}_2 \,.\, \mathrm{X}_1\mathrm{X}_2^2\mathrm{X}_3^2\mathrm{X}_4 = 0,\end{aligned}$$

and the second syzygies

$$\mathrm{X}_1\,(\mathrm{B}_2) - \mathrm{X}_1\mathrm{X}_2\,(\mathrm{B}_1) = 0,$$
$$\mathrm{X}_1\mathrm{X}_2\mathrm{X}_3\,(\mathrm{B}_2) - \mathrm{X}_1\mathrm{X}_2^2\mathrm{X}_3\,(\mathrm{B}_1) = 0.$$

Art. 110. For $s = 5$, the generating function is

$$\underset{\geq}{\Omega}\frac{1}{1 - a_1a_2a_3a_4\mathrm{X}_1 \,.\, 1 - \frac{b_2b_3b_4}{a_1}\mathrm{X}_2 \,.\, 1 - \frac{a_1c_3c_4}{a_2b_2}\mathrm{X}_3 \,.\, 1 - \frac{a_2d_4}{a_1a_3b_3c}\mathrm{X}_4 \,.\, 1 - \frac{a_1b_2a_3}{a_4b_4c_4d_4}\mathrm{X}_5}$$

and there is no difficulty in continuing the series. The obtaining, however, of the reduced forms soon becomes laborious.

Art. 111. There is another method of investigation. Guided by the results obtained let us restrict consideration to the forms

$$\mathrm{X}_1^{\alpha_1}\mathrm{X}_2^{\alpha_2}\ldots\mathrm{X}_s^{\alpha_s}$$

which are such that

$$\alpha_m = \alpha_{s+1-m}\,.\text{*}$$

This is of great importance, because we are thus able, for any given order, to generate the functions of that order alone.

$$\text{Put } \mathrm{X}_m\mathrm{X}_{s+1-m} = \mathrm{Y}_m \text{ and seek } \Sigma\mathrm{Y}_1^{\alpha_1}\mathrm{Y}_2^{\alpha_2}\ldots$$

Art. 112. For $s = 2$, the generating function is simply

$$\frac{1}{1-\mathrm{Y}_1} = \frac{1}{1-\mathrm{X}_1\mathrm{X}_2}\,.$$

Art. 113. For $s = 3$, the conditions

$$2\alpha_1 \geq \alpha \geq \alpha_1$$

lead to

$$\underset{\geq}{\Omega}\frac{1}{1 - \frac{a^2}{b}\mathrm{Y}_1 \,.\, 1 - \frac{b}{a}\mathrm{Y}_2},$$

the letters a, b determining the first and second conditions respectively

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* The validity of this assumption will be considered later.

This is on reduction

$$\frac{1}{1 - Y_1Y_2 \cdot 1 - Y_1Y_2^2} = \frac{1}{1 - X_1X_2X_3 \cdot 1 - X_1X_2^2X_3}$$

a real generating function.

Art. 114. For $s = 4$, the conditions are the same, viz. :—

$$2\alpha_1 \geqq \alpha_2 \geqq \alpha_1$$

and the Ω function, where now

$$Y_1 = X_1X_4, \; Y_2 = X_2X_3,$$

is

$$\underset{>}{\Omega} \frac{1}{1 - \frac{a^2}{b} Y_1 \cdot 1 - \frac{b}{a} Y_2} = \frac{1}{1 - Y_1Y_2 \cdot 1 - Y_1Y_2^2}$$

$$= \frac{1}{1 - X_1X_2X_3X_4 \cdot 1 - X_1X_2^2X_3^2X_4}$$

yielding the ground forms already found by the first method.

Art. 115. For $s = 5$, the conditions are

$$\alpha_1 + \alpha_2 \geqq \alpha_3 \geqq \alpha_2,$$

$$2\alpha_1 \geqq \alpha_2 \geqq \alpha_1;$$

leading to

$$\underset{\geqq}{\Omega} \frac{1}{1 - \frac{ab^2}{d} Y_1 \cdot 1 - \frac{ad}{bc} Y_2 \cdot 1 - \frac{c}{a} Y},$$

where

$$Y_1 = X_1X_5, \; Y_2 = X_2X_4, \; Y_3 = X_3,$$

and this is

$$\underset{\geqq}{\Omega} \frac{1}{1 - \frac{ab^2}{d} Y_1 \cdot 1 - \frac{d}{b} Y_2Y_3 \cdot 1 - \frac{1}{a} Y}$$

$$= \underset{\geqq}{\Omega} \frac{1}{1 - abY_1Y_2Y_3 \cdot 1 - \frac{1}{b} Y_2Y_3 \cdot 1 - \frac{1}{a} Y_3}$$

$$= \underset{\geqq}{\Omega} \frac{1}{1 - bY_1Y_2Y_3 \cdot 1 - \frac{1}{b} Y_2Y_3 \cdot 1 - bY_1Y_2Y_3^2}$$

$$= \frac{1 - Y_1^2Y_2^3Y_3^4}{1 - Y_1Y_2Y_3 \cdot 1 - Y_1Y_2Y_3^2 \cdot 1 - Y_1Y_2^2Y_3^2 \cdot 1 - Y_1Y_2^2Y_3^3}$$

$$= \frac{1 - X_1X_2X_3X_4X_5}{1 - X_1X_2X_3X_4X_5 \cdot 1 - X_1X_2X_3^2X_4X_5 \cdot 1 - X_1X_2^2X_3^2X_4^2X_5 \cdot 1 - X_1X_2^2X_3^3X_4^2X_5},$$

establishing the ground forms

$$X_1X_2X_3X_4X_5, \; X_1X_2X_3^2X_4X_5$$

$$X_1X_2^2X_3^2X_4^2X_5, \; X_1X_2^2X_3^3X_4^2X_5$$

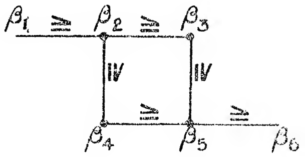
connected by the simple syzygy

$$(X_1X_2X_3X_4X_5)\,(X_1X_2^2X_3^3X_4^2X_5) - (X_1X_2X_3^2X_4X_5)\;X_1X_2^2X_3^2X_4^2X_5) = 0.$$

Art. 116. I stop to remark that one of these ground forms, viz.:—

$$X_1X_2X_3^2X_4X_5$$

is new, not having so far presented itself in a partition theorem. It is one of an infinite system which merits, and will receive, separate consideration later on. The one before us is associated with partitions at the points of the dislocated lattice.



Art. 117 For $s = 6$, the conditions are :

$$2\alpha_2 \geqq \alpha_1 + \alpha_3$$
$$2\alpha_1 \geqq \alpha_2$$
$$\alpha_3 \geqq \alpha_2,$$

leading to

$$\underset{\geqq}{\Omega}\;\frac{1}{1-\frac{b^2}{a}Y_1\,.\,1-\frac{a^2}{bc}Y_2\,.\,1-\frac{c}{a}Y_3},$$

where

$$Y_1 = X_1X_6,\quad Y_2 = X_2X_5,\quad Y_3 = X_3X_4.$$

This is

$$\underset{\geqq}{\Omega}\;\frac{1}{1-\frac{b^2}{a}Y_1\,.\,1-\frac{a}{b}Y_2Y_3\,.\,1-\frac{1}{a}Y}$$

$$= \underset{\geqq}{\Omega}\;\frac{1}{1-\frac{1}{a}Y_3\,.\,1-\frac{a}{b}Y_2Y_3\,.\,1-bY_1Y_2Y_3}$$

$$= \frac{1}{1-Y_1Y_2Y_3\,.\,1-Y_1Y_2^2Y_3^2\,.\,1-Y_1Y_2^2Y^3},$$

establishing the ground forms :

$$X_1X_2X_3X_4X_5X_6$$
$$X_1X_2^2X_3^2X_4^2X_5^2X_6$$
$$X_1X_2^2X_3^3X_4^3X_5^2X_6,$$

unconnected by any syzygy.

Art. 118. For $s = 7$, the independent conditions are:

$$\begin{aligned} 2\alpha_1 + 2\alpha_3 &\geqq 2\alpha_2 + \alpha_4 \\ 2\alpha_2 &\geqq \alpha_4 \\ \alpha_1 + \alpha_2 &\geqq \alpha_3 \\ \alpha_4 &\geqq \alpha_3 \\ 2\alpha_1 &\geqq \alpha_2 \\ \alpha_3 &\geqq \alpha_2 \\ \alpha_2 &\geqq \alpha_1, \end{aligned}$$

and these lead to

$$\underset{\geqq}{\Omega} \frac{}{1 - \frac{a^2ce^2}{g}\mathrm{Y}_1 . 1 - \frac{b^2cg}{a^2ef}\mathrm{Y}_2 . 1 - \frac{a^2f}{cd}\mathrm{Y}_3 . 1 - \frac{d}{ab}\mathrm{Y}_4}$$

and eliminating d, f, g in succession

$$= \underset{\geqq}{\Omega} \frac{1}{1 - \frac{a^2ce^2}{g}\mathrm{Y}_1 . 1 - \frac{b^2cg}{a^2ef}\mathrm{Y}_2 . 1 - \frac{af}{bc}\mathrm{Y}_3\mathrm{Y}_4 . 1 - \frac{1}{ab}\mathrm{Y}_4}$$

$$= \underset{\geqq}{\Omega} \frac{1}{1 - \frac{a^2ce^2}{g}\mathrm{Y}_1 . 1 - \frac{bg}{ae}\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4 . 1 - \frac{a}{bc}\mathrm{Y}_3\mathrm{Y}_4 . 1 - \frac{1}{ab}\mathrm{Y}_4},$$

$$= \underset{\geqq}{\Omega} \frac{1}{1 - abce\mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4 . 1 - \frac{b}{ae}\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4 . 1 - \frac{a}{bc}\mathrm{Y}_3\mathrm{Y}_4 . 1 - \frac{1}{ab}\mathrm{Y}_4}$$

and eliminating e

$$= \underset{\geqq}{\Omega} \frac{1}{1 - abc\mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4 . 1 - b^2c\mathrm{Y}_1\mathrm{Y}_2^2\mathrm{Y}_3^2\mathrm{Y}_4^2 . 1 - \frac{a}{bc}\mathrm{Y}_3\mathrm{Y}_4 . 1 - \frac{1}{ab}\mathrm{Y}}$$

and eliminating c

$$= \underset{\geqq}{\Omega} \frac{1 - a^2b^2\mathrm{Y}_1^2\mathrm{Y}_2^3\mathrm{Y}_3^4\mathrm{Y}_4^4}{\begin{array}{c} 1 - ab\mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4 . 1 - a^2\mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3^2\mathrm{Y}_4^2 . 1 - b^2\mathrm{Y}_1\mathrm{Y}_2^2\mathrm{Y}_3^2\mathrm{Y}_4^2, \\ 1 - ab . \mathrm{Y}_1\mathrm{Y}_2^2\mathrm{Y}_3^3\mathrm{Y}_4^3 . 1 - \frac{1}{ab}\mathrm{Y}_4 . \end{array}}$$

And on further reduction it is finally

$$\frac{1 - \mathrm{Y}_1^2\mathrm{Y}_2^3\mathrm{Y}_3^4\mathrm{Y}_4^4 . 1 - \mathrm{Y}_1^2\mathrm{Y}_2^3\mathrm{Y}_3^4\mathrm{Y}^5}{\begin{array}{c} 1 - \mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4 . 1 - \mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4^2 . 1 - \mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3^2\mathrm{Y}_4^2, \\ 1 - \mathrm{Y}_1\mathrm{Y}_2^2\mathrm{Y}_3^2\mathrm{Y}_4^2 . 1 - \mathrm{Y}_1\mathrm{Y}_2^2\mathrm{Y}_3^3\mathrm{Y}_4^3 . 1 - \mathrm{Y}_1\mathrm{Y}_2^2\mathrm{Y}_3^3\mathrm{Y}_4^4 \end{array}}$$

establishing the ground forms

$$\begin{aligned}
Y_1Y_2Y_3Y_4 &\equiv X_1X_2X_3X_4X_5X_6X_7\\
Y_1Y_2Y_3Y_4^2 &\equiv X_1X_2X_3X_4^2X_5X_6X_7\\
Y_1Y_2Y_3^2Y_4^2 &\equiv X_1X_2X_3^2X_4^2X_5^2X_6X_7\\
Y_1Y_2^2Y_3^2Y_4^2 &\equiv X_1X_2^2X_3^2X_4^2X_5^2X_6^2X_7\\
Y_1Y_2^2Y_3^3Y_4^3 &\equiv X_1X_2^2X_3^3X_4^3X_5^3X_6^2X_7\\
Y_1Y_2^2Y_3^3Y_4^4 &\equiv X_1X_2^2X_3^3X_4^4X_5^3X_6^2X_7
\end{aligned}$$

connected by the simple syzygies

$$(Y_1Y_2Y_3Y_4)\,(Y_1Y_2^2Y_3^3Y_4^3) - (Y_1Y_2Y_3^2Y_4^2)\,(Y_1Y_2^2Y_3^2Y_4^2) = 0,$$
$$(Y_1Y_2Y_3Y_4)\,(Y_1Y_2^2Y_3^3Y_4^4) - (Y_1Y_2Y_3Y_4^2)\,(Y_1Y_2^2Y_3^3Y_4^3) = 0,$$

and, denoting these respectively by A and B, the numerator term $+\,Y_1^4Y_2^6Y_3^8Y_4^9$ indicates the second, or compound, syzygy :—

$$(Y_1Y_2Y_3Y_4)\,(Y_1Y_2^2Y_3^3Y_4^4)\,(A) - (Y_1Y_2Y_3^2Y_4^2)\,(Y_1Y_2^2Y_3^2Y_4^2)\,(B) = 0.$$

Art. 119. I remark that the forms

$$Y_1Y_2Y_3Y_4^2, \qquad Y_1Y_2Y_3^2Y_4^2$$

are new to partition theory.

Art. 120. For $s = 8$, the reduced conditions are

$$\begin{aligned}
\alpha_2 + \alpha_3 &\geqq \alpha_1 + \alpha_4\\
\alpha_1 + \alpha_2 &\geqq \alpha_3\\
\alpha_4 &\geqq \alpha_3\\
2\alpha_1 &\geqq \alpha_2\\
\alpha_3 &\geqq \alpha_2
\end{aligned}$$

leading to

$$\underset{\geqq}{\Omega}\,\frac{1}{1-\frac{bd^2}{a}Y_1\,.\,1-\frac{ab}{de}Y_2\,.\,1-\frac{ae}{bc}Y_3\,.\,1-\frac{c}{a}Y_4}$$

$$=\underset{\geqq}{\Omega}\,\frac{1}{1-\frac{bd^2}{a}Y_1\,.\,1-\frac{a}{d}Y_2Y_3Y_4\,.\,1-\frac{1}{b}Y_3Y_4\,.\,1-\frac{1}{a}Y_4}$$

$$=\underset{\geqq}{\Omega}\,\frac{1}{1-bdY_1Y_2Y_3Y_4\,.\,1-\frac{1}{d}Y_2Y_3Y_4\,.\,1-\frac{1}{b}Y_3Y_4\,.\,1-\frac{1}{d}Y_2Y_3Y_4^2}$$

$$=\underset{\geqq}{\Omega}\,\frac{1}{1-bY_1Y_2Y_3Y_4\,.\,1-bY_1Y_2^2Y_3^2Y_4^2\,.\,1-\frac{1}{b}Y_3Y_4\,.\,1-bY_1Y_2^2Y_3^2Y_4^3}$$

$$=\frac{1-Y_1^2Y_2^3Y_3^4Y_4^4-Y_1^2Y_2^3Y_3^4Y_4^5-Y_1^2Y_2^4Y_3^5Y_4^6+Y_1^3Y_2^5Y_3^6Y_4^7+Y_1^3Y_2^5Y_3^7Y_4^8}{\begin{array}{c}1-Y_1Y_2Y_3Y_4\,.\,1-Y_1Y_2Y_3^2Y_4^2\,.\,1-Y_1Y_2^2Y_3^2Y_4^2\\ 1-Y_1Y_2^2Y_3^2Y_4^3\,.\,1-Y_1Y_2^2Y_3^3Y_4^3\,.\,1-Y_1Y_2^2Y_3^3Y_4^4\end{array}}$$

indicating the ground forms

$$
\begin{aligned}
Y_1Y_2Y_3Y_4 &\equiv X_1X_2X_3X_4X_5X_6X_7X_8\\
Y_1Y_2Y_3^2Y_4^2 &\equiv X_1X_2X_3^2X_4^2X_5^2X_6^2X_7X_8\\
Y_1Y_2^2Y_3^2Y_4^2 &\equiv X_1X_2^2X_3^2X_4^2X_5^2X_6^2X_7^2X_8\\
Y_1Y_2^2Y_3^2Y_4^3 &\equiv X_1X_2^2X_3^2X_4^3X_5^3X_6^2X_7^2X_8\\
Y_1Y_2^2Y_3^3Y_4^3 &\equiv X_1X_2^2X_3^3X_4^3X_5^3X_6^3X_7^2X_8\\
Y_1Y_2^2Y_3^3Y_4^4 &\equiv X_1X_2^2X_3^3X_4^4X_5^4X_6^3X_7^2X_8
\end{aligned}
$$

Art. 121. So far it appears that all products which can be placed in the form of a rectangle

$$
\begin{array}{lllll}
X_1 & X_2 & X_3 & \ldots & X_l\\
X_2 & X_3 & X_4 & \ldots & X_{l+1}\\
\vdots & \vdots & \vdots & & \vdots\\
X_m & X_{m+1} & X_{m+2} & \ldots & X_{l+m-1}
\end{array}
$$

are ground forms for all values of l and m.

I have established this independently, and thus proved that the conjectured result for the general lattice *in plano* is, at any rate, finite and integral, as it should be.

It is desirable to obtain information concerning the ground forms which are not within the rectangular tableau.

The forms

$$X_1^{\alpha_1}X_2^{\alpha_2}\ldots X_s^{\alpha_s},$$

which appear in the tableau, may be eliminated from consideration, with the exception of the form

$$X_1X_2\ldots X_s,$$

by ascribing additional conditions such as

$$\alpha_1 = \alpha_2,$$

which are not true in the tableau.

The condition of this tableau is that if $\alpha_p = \alpha_{p+1}$, no index α_{p+q} is greater than α_p; after a repetition of index, no rise in index takes place. In the Y form, therefore, we may assign the conditions

$$\alpha_p = \alpha_{p+1} < \alpha_{p+2}$$

for any value of p, as one excluding the whole of the forms appertaining to the tableau.

We may impress the conditions

$$
\begin{aligned}
\alpha_1 &= \alpha_2 < \alpha_3\\
\alpha_2 &= \alpha_3 < \alpha_4\\
\alpha &= \alpha_4 < \alpha_5\\
&\ldots
\end{aligned}
$$

in succession, and we may combine any number of such conditions as are independent.

Art. 122. I postpone further investigation into this interesting theory, and will now give a formal proof that the product tableau is, in fact, finite and integral. The product in question for $m \geqq l$ is

$$X_1 X_2^2 \ldots X_{l-1}^{l-1} (X_l X_{l+1} \ldots X_m)^l X_{m+1}^{l-1} X_{m+2}^{l-2} \ldots X_{m+l-2}^2 X_{m+l-1}$$

so that

$$\begin{array}{lll} \alpha_s = s & \text{for} & l \geqq s \\ \alpha_s = l & \text{for} & s > l \text{ and } < m+1 \\ \alpha_{m+s} = l - s & \text{for} & l - 1 \geqq s \end{array}$$

All the conditions may be resumed in the single formula

$$\alpha_s + \alpha_{2s+t} + \alpha_{3s+2t} + \ldots \geqq \alpha_{s+t} + \alpha_{2s+2t} + \alpha_{3s+3t} + \ldots$$

s and t being any integers.

Let the greatest integer in $\frac{l+t-1}{s+t}$ be denoted by $I_1 \frac{l+t-1}{s+t}$ or by I_1 simply for brevity. Similarly let I_2 refer to $\frac{m+t}{s+t}$, I_3 to $\frac{l+m+t-1}{s+t}$, J_1 to $\frac{l-1}{s+t}$, J_2 to $\frac{m}{s+t}$, and J_3 to $\frac{l+m-1}{s+t}$. We derive

$$\begin{array}{rcl} I_1 = J_1 & \text{or} & J_1 + 1 \\ I_2 = J_2 & \text{or} & J_2 + 1 \\ I_3 = J_3 & \text{or} & J_3 + 1 \\ I_1 + I_2 = I_3 & \text{or} & I_3 + 1 \\ J_1 + J_2 = J_3 & \text{or} & J_3 - 1 \end{array}$$

and we have ten possible cases to consider, viz. :—

Case 1.

$$\begin{array}{ll} I_1 = J_1 & I_1 + I_2 = I_3 \\ I_2 = J_2 & J_1 + J_2 = J_3 \\ I_3 = J_3 & \end{array}$$

Case 2.

$$\begin{array}{ll} I_1 = J_1 + 1 & I_1 + I_2 = I_3 \\ I_2 = J_2 & J_1 + J_2 = J_3 - 1 \\ I_3 = J_3 & \end{array}$$

Case 3.

$$\begin{array}{ll} I_1 = J_1 + 1 & I_1 + I_2 = I_3 + 1 \\ I_2 = J_2 & J_1 + J_2 = J_3 \\ I_3 = J_3 & \end{array}$$

Case 4.

$$\begin{array}{ll} I_1 = J_1 & I_1 + I_2 = I_3 \\ I_2 = J_2 + 1 & J_1 + J_2 = J_3 - 1 \\ I_3 = J_3 & \end{array}$$

Case 5.

$$\begin{array}{ll} I_1 = J_1 & I_1 + I_2 = I_3 + 1 \\ I_2 = J_2 + 1 & J_1 + J_2 = J_3 \\ I_3 = J_3 & \end{array}$$

Case 6.

$$\begin{array}{ll} I_1 = J_1 + 1 & I_1 + I_2 = I_3 + 1 \\ I_2 = J_2 + 1 & J_1 + J_2 = J_3 - 1 \\ I_3 = J_3 & \end{array}$$

Case 7.

$$\begin{array}{ll} I_1 = J_1 + 1 & I_1 + I_2 = I_3 \\ I_2 = J_2 & J_1 + J_2 = J_3 \\ I_3 = J_3 + 1 & \end{array}$$

Case 8.

$$\begin{array}{ll} I_1 = J_1 & I_1 + I_2 = I_3 \\ I_2 = J_2 + 1 & J_1 + J_2 = J_3 \\ I_3 = J_3 + 1 & \end{array}$$

Case 9.

$$\begin{array}{ll} I_1 = J_1 + 1 & I_1 + I_2 = I_3 \\ I_2 = J_2 + 1 & J_1 + J_2 = J_3 - 1 \\ I_3 = J_3 + 1 & \end{array}$$

Case 10.

$$\begin{array}{ll} I_1 = J_1 + 1 & I_1 + I_2 = I_3 + 1 \\ I_2 = J_2 + 1 & J_1 + J_2 = J_3 \\ I_3 = J_3 + 1 & \end{array}$$

For the series

$$\alpha_s + \alpha_{2s+t} + \alpha_{3s+2t} + \ldots,$$

we have, as far as α_{l-1}, I_1 terms; as far as α_m, I_2 terms; and, as far as α_{l+m-1}, I_3 terms.

Hence the summation gives :—

$$\begin{aligned} &\tfrac{1}{2} I_1 \{2s + (I_1 - 1)(s + t)\} + l(I_2 - I_1) \\ &\quad + \tfrac{1}{2}(I_3 - I_2)\{2l + 2m + 2t - 2(s + t)(I_2 + 1) - (s + t)(I_3 - I_2 - 1)\} \\ &= \tfrac{1}{2}(s + t)(I_1^2 + I_2^2 - I_3^2) + (\tfrac{1}{2}s - \tfrac{1}{2}t - \tfrac{1}{2}l) I_1 \\ &\quad + (\tfrac{1}{2}s - \tfrac{1}{2}t - m) I_2 + (l + m - \tfrac{1}{2}s + \tfrac{1}{2}t) I_3. \end{aligned}$$

Summing similarly the series

$$a_{s+t} + a_{2s+2t} + a_{3s+3t} \dots$$

we find

$$\tfrac{1}{2}(s+t)(\mathrm{J}_1^2 + \mathrm{J}_2^2 - \mathrm{J}_3^2) + (\tfrac{1}{2}s + \tfrac{1}{2}t - l)\,\mathrm{J}_1 \\ + (\tfrac{1}{2}s + \tfrac{1}{2}t - m)\,\mathrm{J}_2 + (l + m - \tfrac{1}{2}s - \tfrac{1}{2}t)\,\mathrm{J}_3,$$

and we have in each of the ten cases to establish the relation

$$\tfrac{1}{2}(s+t)(\mathrm{I}_1^2 + \mathrm{I}_2^2 - \mathrm{I}_3^2) + (\tfrac{1}{2}s - \tfrac{1}{2}t - l)\,\mathrm{I}_1 \\ + (\tfrac{1}{2}s - \tfrac{1}{2}t - m)\,\mathrm{I}_2 + (l + m - \tfrac{1}{2}s + \tfrac{1}{2}t)\,\mathrm{I}_3, \\ \geqq \tfrac{1}{2}(s+t)(\mathrm{J}_1^2 + \mathrm{J}_2^2 - \mathrm{J}_3^2) + (\tfrac{1}{2}s + \tfrac{1}{2}t - l)\,\mathrm{J}_1 \\ + (\tfrac{1}{2}s + \tfrac{1}{2}t - m)\,\mathrm{J}_2 + (l + m - \tfrac{1}{2}s - \tfrac{1}{2}t)\,\mathrm{J}_3$$

for all values of s and t.

For Case 1 it reduces to

$$\mathrm{I}_1 + \mathrm{I}_2 \geqq \mathrm{I}_3,$$

which is true, for here $\mathrm{I}_1 + \mathrm{I}_2 = \mathrm{I}_3$.

For Case 2, making use of $\mathrm{J}_1 + \mathrm{J}_2 = \mathrm{J}_3 - 1$, the reduction is to

$$\mathrm{J}_1 \geqq \frac{l - s - t}{s + t},$$

and J_1 being the greatest integer in $\frac{l-1}{s+t}$, and moreover $s + t$ being at least unity, the relation is obviously satisfied.

For Case 3, making use of $\mathrm{J}_1 + \mathrm{J}_2 = \mathrm{J}_3$, we find

$$\mathrm{J}_1 \geqq \frac{l - s}{s + t},$$

and this is satisfied as $s \geqq 1$.

For Case 4, reducing by $\mathrm{J}_1 + \mathrm{J}_2 = \mathrm{J}_3 - 1$, we find

$$\mathrm{J}_2 \geqq \frac{m}{s+t} - 1$$

obviously true from the definition of J_2.

For Case 5, reducing by $\mathrm{J}_1 + \mathrm{J}_2 = \mathrm{J}_3$, we find

$$\mathrm{J}_2 \geqq \frac{m - s}{s + t}$$

obviously satisfied.

For Case 6, reducing by $\mathrm{J}_1 + \mathrm{J}_2 = \mathrm{J}_3 - 1$, we find

$$\mathrm{J}_3 \geqq \frac{l + m - s}{s + t}$$

clearly satisfied.

For Case 7, reducing by $J_1 + J_2 = J_3$, we find

$$J_2 \geq \frac{m}{s+t},$$

which is right.

For Case 8, reducing by $J_1 + J_2 = J_3$, we find

$$J_1 \geq \frac{l}{s+t},$$

which is satisfied.

For Case 9, reducing by $J_1 + J_2 = J_3 - 1$, the ratio is one of equality.

For Case 10, reducing by $J_1 + J_2 = J_3$, we find

$$s + t \geq 0,$$

which is right.

Hence the relation is universally satisfied, and we have proved that the expression

$$X_1 X_2^2 \ldots X_{l-1}^{l-1} (X_l X_{l+1} \ldots X_m)^l X_{m+1}^{l-1} X_{m+2}^{l-2} \ldots X_{l+m-1}$$

is in every case finite and integral.

Art. 123. In Part 3 of this Memoir I hope to treat of other systems of algebraical and arithmetical functions which fall within the domain of partition analysis and the theory of the linear composition of integers; also to take up the general theories of partition analysis and linear Diophantine analysis, with possible extensions to higher degrees.